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The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

mathematical models and methods

Unit 19
Numerical methods for differential equations

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Numerical methods for differential equations

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1.8

Contents

Introduction

This unit is devoted to the study of numerical methods for finding approximate solutions of differential equations. Only first-order differential equations of the form

$$y' = m(x, y),$$

with $y(x_0)$ given, will be considered but the methods in this unit can be adapted to find approximate solutions of higher-order differential equations. (A differential equation of any order may be treated as a system of first-order differential equations, as described in Section 5 of *Unit 6*.)

The primary objective of this unit is to enable you to use numerical methods intelligently to obtain approximate solutions. To do this it is necessary to study several different methods

(i) to see how they work,
(ii) to determine whether one method is better than another,
(iii) to see when things go wrong.

All the methods in this text lead to recurrence relations which can be used step by step to obtain an approximate solution at equally spaced points $x_0, x_1, x_2, \ldots$. As in *Unit 18* we denote the values of this approximate solution by $Y_0, Y_1, Y_2 \ldots$. It is important to compare this solution with the true solution $y = y(x)$. In particular we are interested in the difference between the approximate value, Y_n, and the true value, y_n, at x_n.

Study guide

This unit is intended to be read sequentially. In Sections 1 and 2 we derive some numerical methods and use them to solve simple problems. The television section (Section 2) illustrates some of these methods and how they can be used to solve the logistic equation. Before viewing the programme it is advisable to have read Subsection 1.1 on the trapezoidal method and also the pre-broadcast notes.

In Section 3 and in the tape session in Section 4 we look at the methods of the first two sections to discover why they work, when they go wrong and what sort of accuracy you could expect.

In the remainder of Section 4 we look at Simpson's method which is more accurate than most of the one-step methods but can be difficult to use.

Section 5 describes a computer package which you can use to solve differential equations numerically. One of the TMA questions for this unit may well require you to use this package.

You can use the computer package at Summer School.

1 Numerical methods

1.1 Integration methods

In *Unit 18* you met three integration methods for evaluating the integral

$$I = \int_{x_r}^{x_{r+1}} f(x)\, dx.$$

They were Euler's method, the trapezoidal method and Simpson's method. In this section we will see how the first two of these methods can be used to solve first-order differential equations of the form

$$y' = m(x, y). \qquad (1)$$

y' is short for $\dfrac{dy}{dx}$.

In Section 4 we will examine Simpson's method.

If we integrate both sides of Equation (1) between x_r and x_{r+1} we have

$$\int_{x_r}^{x_{r+1}} y'\, dx = \int_{x_r}^{x_{r+1}} m(x, y)\, dx. \qquad (2)$$

The left-hand side of this equation can be integrated to give

$$\int_{x_r}^{x_{r+1}} y'\,dx = y_{r+1} - y_r.$$

y_r is short for $y(x_r)$.

Hence we may write Equation (2) as

$$y_{r+1} - y_r = \int_{x_r}^{x_{r+1}} m(x,y)\,dx. \qquad (3)$$

Thus, if we can determine the integral on the right-hand side and if we know y_r, we can compute y_{r+1} using Equation (3). The difficulty with this approach is that we do not know the function y, so it is unlikely that we will be able to determine the integral.

For example, the differential equation

$$y' = xy \qquad \text{with } y(0) = 0$$

has $m(x,y) = xy$.

Equation (3) becomes

$$y_{r+1} = y_r + \int_{x_r}^{x_{r+1}} xy\,dx.$$

Since we do not know $y(x)$ we cannot proceed to determine an exact recurrence relation, but we can determine an approximate one if we approximate the integral.

In the last unit we saw that the simplest method for approximating integrals was Euler's method (see Figure 1) giving

$$\int_{x_r}^{x_{r+1}} f(x)\,dx \simeq hf(x_r)$$

where $h = x_{r+1} - x_r$.

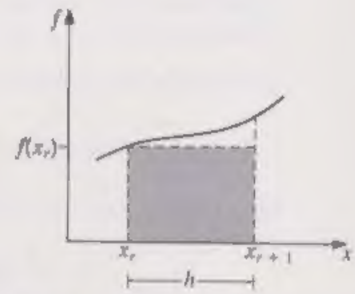


Figure 1. The shaded area is Euler's approximation to the area under the curve.

In the same way we can approximate the integral in Equation (3) using Euler's method as

$$\int_{x_r}^{x_{r+1}} m(x,y)\,dx \simeq hm(x_r, y_r).$$

With this approximation Equation (3) becomes

$$Y_{r+1} = Y_r + hm(x_r, Y_r).$$

(I have used capital letters, Y_r and Y_{r+1}, here because Y_r will only be an approximation to the true solution.) But this recurrence relation is just Euler's method for solving a first-order differential equation, as described in *Units 2* and *18*. This new derivation illustrates how integration methods can be used to solve differential equations.

To simplify the notation, we define

$$Y'_r = m(x_r, Y_r)$$

and write the recurrence relation for Euler's method as

$$Y_{r+1} = Y_r + hY'_r.$$

Euler's method

1. You are given a differential equation

$$y' = m(x, y)$$

and a value for $y(x_0)$.

2. To apply Euler's method, choose a step-size, h, and calculate $Y_1, Y_2 \ldots$ from

$$Y_{r+1} = Y_r + hY'_r$$

where $Y'_r = m(x_r, Y_r)$,

$Y_0 = y(x_0)$,

and $x_r = x_0 + rh$.

3. Y_r is an approximation to y_r.

Another integration method is the trapezoidal method. From the results in *Unit 18* it is clear that this method for approximating the integral by a trapezium (see Figure 2) as

$$\int_{x_r}^{x_{r+1}} f(x)\,dx \simeq \frac{h}{2}[f(x_r) + f(x_{r+1})]$$

is generally more accurate than Euler's method. We would thus expect to get more accurate results if we use this method to solve differential equations.

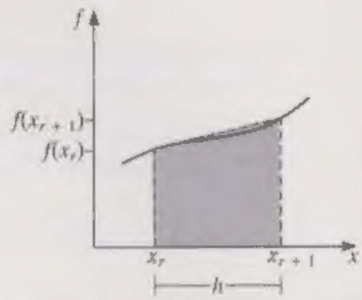


Figure 2. The shaded area is the trapezoidal approximation to the area under the curve.

If we apply this method to approximate the integral in Equation (3) we have

$$\int_{x_r}^{x_{r+1}} m(x, y)\,dx \simeq \frac{h}{2}[m(x_r, y_r) + m(x_{r+1}, y_{r+1})].$$

This gives rise to the recurrence relation

$$Y_{r+1} = Y_r + \frac{h}{2}[m(x_r, Y_r) + m(x_{r+1}, Y_{r+1})]$$

or, more simply,

$$Y_{r+1} = Y_r + \frac{h}{2}[Y'_r + Y'_{r+1}]. \qquad (4)$$

However there is a difficulty in using the trapezoidal method because Equation (4) contains a term in Y'_{r+1} $(= m(x_{r+1}, Y_{r+1}))$ on the right-hand side. We do not know this value until we have computed Y_{r+1}. Equation (4) would have to be solved for Y_{r+1}. For *linear* equations this can be done fairly easily, as Example 1 shows.

Example 1

Use the trapezoidal method with $h = 0.1$ to determine the approximate solution for $x_r = rh$, $0 \le r \le 10$, of

$$y' = x + y \qquad \text{with } y(0) = 0.$$

Compare the answers with the true solution, $y = e^x - x - 1$.

Solution

We want to apply the recurrence relation (4). The differential equation tells us that

$$m(x, y) = x + y$$

and so $Y'_r = x_r + Y_r$

and $Y'_{r+1} = x_{r+1} + Y_{r+1}$.

We substitute for Y'_r and Y'_{r+1} in Equation (4) to get

$$Y_{r+1} = Y_r + \frac{h}{2}[(x_r + Y_r) + (x_{r+1} + Y_{r+1})].$$

To solve for Y_{r+1} we collect up the terms in Y_{r+1} and Y_r as

$$Y_{r+1}\left(1-\frac{h}{2}\right) = Y_r\left(1+\frac{h}{2}\right) + \frac{h}{2}(x_r + x_{r+1}).$$

i.e. $$Y_{r+1} = \frac{(1+h/2)Y_r + (h/2)(x_r + x_{r+1})}{1-h/2}$$

We can use this recurrence relation to generate the approximate solution. We have

(i) $h = 0.1$,
(ii) $x_r = x_0 + rh = 0.1r$.

We can simplify the recurrence relation to

$$Y_{r+1} = \frac{1.05Y_r + 0.05(0.1r + 0.1(r+1))}{0.95}$$

i.e. $$Y_{r+1} = \frac{210Y_r + 2r + 1}{190}.$$

This can be expressed as

$$Y_{r+1} = \frac{21}{19}Y_r + \frac{r}{95} + \frac{1}{190}$$

which is a linear recurrence relation.

The computation is now straightforward, starting with $Y_0 = 0$. The following table shows how the approximate solution, generated using the recurrence relation for the trapezoidal method, compares with the true solution. The results for Euler's method and the Taylor series method of order 2 are also given for comparison. (These were obtained in *Unit 18*.)

Solutions to $y' = x + y$ with $y(0) = 0$, $h = 0.1$.

r	x_r	trapezoidal method	true solution	Euler's method	Taylor series method of order 2
0	0.0	0.0000	0.0000	0.0000	0.0000
1	0.1	0.0053	0.0052	0.0000	0.0050
2	0.2	0.0216	0.0214	0.0100	0.0210
3	0.3	0.0502	0.0499	0.0310	0.0492
4	0.4	0.0923	0.0918	0.0641	0.0909
5	0.5	0.1494	0.1487	0.1105	0.1474
6	0.6	0.2230	0.2221	0.1716	0.2204
7	0.7	0.3149	0.3138	0.2487	0.3116
8	0.8	0.4270	0.4255	0.3436	0.4228
9	0.9	0.5615	0.5596	0.4579	0.5562
10	1.0	0.7206	0.7183	0.5937	0.7141

By comparing columns 3 and 4 we see that the results for the trapezoidal method are reasonably accurate although, as expected, the error increases as x increases. The approximate value of y at $x = 1$ is correct to two decimal places. Now compare the true solution with the results obtained using Euler's method. For this problem Euler's method gives much worse results and a much smaller step-size would be needed to obtain reasonable accuracy. On the other hand the results for the Taylor series method of order 2 are about as accurate as the results for the trapezoidal method.

The manipulation required to use the trapezoidal method to solve the general linear differential equation of the form

$$y' = l(x)y + k(x), \tag{5}$$

using a step-size h, can be deduced in the same way as in Example 1. The recurrence relation (4) for the trapezoidal method is

$$Y_{r+1} = Y_r + \frac{h}{2}(Y'_r + Y'_{r+1}). \tag{6}$$

Using Equation (5) we have

$$Y'_r = l_r Y_r + k_r$$

and $$Y'_{r+1} = l_{r+1}Y_{r+1} + k_{r+1}.$$

l_r is short for $l(x_r)$, k_r is short for $k(x_r)$ and so on.

Substituting for Y'_r and Y'_{r+1} in Equation (6) gives

$$Y_{r+1} = Y_r + \frac{h}{2}[(l_r Y_r + k_r) + (l_{r+1} Y_{r+1} + k_{r+1})]$$

which can be solved for Y_{r+1} as

$$Y_{r+1} = \frac{[1 + (h/2)l_r]Y_r + (h/2)(k_r + k_{r+1})}{1 - (h/2)l_{r+1}}.$$

The trapezoidal method for linear differential equations

1. You are given a linear differential equation

$$y' = l(x)y + k(x)$$

and a value for $y(x_0)$.

2. To apply the trapezoidal method, choose a step size h and calculate $Y_1, Y_2, \ldots$ from

$$Y_{r+1} = \frac{[1 + (h/2)l_r]Y_r + (h/2)(k_r + k_{r+1})}{1 - (h/2)l_{r+1}}$$

where $Y_0 = y(x_0)$, and $x_r = x_0 + rh$.

3. Y_r is an approximation to y_r.

We have seen two methods in this section: Euler's method, in which the recurrence relation can be applied directly, and the trapezoidal method, which is difficult to use except for linear equations. The difference between the two methods, which makes the trapezoidal method harder to use, can be described by saying that the trapezoidal method is *implicit* whereas Euler's is *explicit*.

In general any numerical method for solving differential equations in which the recurrence relation involves Y'_{r+1} or higher derivatives at x_{r+1} is called **implicit**. This usually implies that we cannot write down the value of Y_{r+1} directly. On the other hand a method is called **explicit** if Y'_{r+1} or higher derivatives at x_{r+1} do not appear in the recurrence relation for Y_{r+1}. In this case Y_{r+1} is defined explicitly in terms of previous values.

Exercise 1

Use the trapezoidal method with $h = 0.2$ to approximate the solution to the differential equation

$$y' = 3y + \sin x \qquad \text{with } y(0) = 0,$$

at $x = 0.2$, 0.4 and 0.6.

The correct solution is

$$y = \frac{e^{3x}}{10} - \frac{3}{10}\sin x - \frac{1}{10}\cos x$$

which gives $y(0.2) = 0.02460$, $y(0.4) = 0.12308$ and $y(0.6) = 0.35304$.

Compare the approximate solution with the true solution by calculating the errors.

Exercise 2

Write down the formula to be used when the trapezoidal method with $h = 0.1$ is to be applied to the differential equation

$$y' = \sin y \qquad \text{with } y(0) = 1.$$

What is the difficulty with this method if you try to use the recurrence relation to generate the approximate solution?

[*Solutions on p. 50*]

1.2 Local truncation errors

So far in this course you have met the Taylor series methods (of which Euler's method is one) and the integration methods (of which Euler's method is again one). By the end of the unit you will be familiar with two more methods. With so

many methods available what is the best method to use? This question is by no means easy to answer and I shall devote most of Section 3 to examining some of the pitfalls. For the moment we will assume that we just have the choice between the Taylor series methods and the trapezoidal method and want to make some simple comparison between them.

Ideally for a given problem we would like to compare the error in the nth term in the sequence of approximations generated by method A with the error in the same term given by method B.

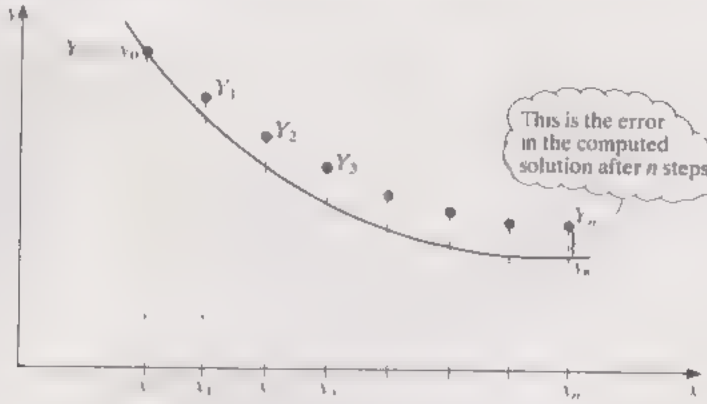


Figure 3. Graph of the approximate solution and the true solution

The point (x_n, y_n) has, for simplicity, been labelled y_n and so on

However, to do this we would need to know the true solution, which is not normally available, so we need an alternative method of comparison. There is a fairly simple yardstick by which we can judge a numerical method and that is to estimate the error introduced in one single step; if we use a method which introduces a comparatively small error at each step then the overall error is likely to be comparatively small.

From *Unit 2* Section 1 on direction fields we know that there is a whole family of solution curves. Thus after r steps we can assume that our approximation Y_r lies on some solution curve (not necessarily the one defined by the initial condition). The error we are going to measure is the distance of Y_{r+1} from this particular solution curve (see Figure 4).

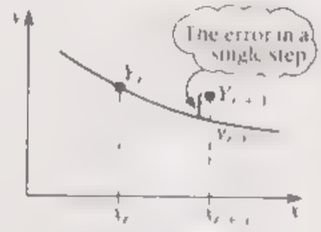


Figure 4. Graph showing the error in a single step.

To illustrate the technique we look at Euler's method, given by

$$Y_{r+1} = Y_r + hY'_r.$$

Let $y = y(x)$ be the solution curve in Figure 4 on which Y_r lies. We have

(i) $Y_r = y_r$,

(ii) $Y'_r = m(x_r, Y_r) = m(x_r, y_r) = y'_r$.

Thus

$$Y_{r+1} = y_r + hy'_r. \tag{7}$$

The true value, y_{r+1}, can be found by using the Taylor series expansion at x_r to give

$$y_{r+1} = y_r + hy'_r + \frac{h^2}{2}y''_r + \frac{h^3}{6}y'''_r + \cdots . \tag{8}$$

The error in Y_{r+1} is given by $Y_{r+1} - y_{r+1}$ which we can write, using Equations (7) and (8), as

$$\begin{aligned} Y_{r+1} - y_{r+1} &= (y_r + hy'_r) - \left(y_r + hy'_r + \frac{h^2}{2}y''_r + \frac{h^3}{6}y'''_r + \cdots\right) \\ &= -\frac{h^2}{2}y''_r - \frac{h^3}{6}y'''_r - \cdots . \end{aligned} \tag{9}$$

This error in Y_{r+1} is called the **local truncation error** because it is

(i) *local* in the sense that we are looking at the error in a single step, i.e. in the locality of x_r;

(ii) a *truncation* error in the following sense: by comparing Equations (7) and (8) it can be seen that we could have obtained the value of Y_{r+1} by truncating the Taylor series in Equation (8).

Look at the local truncation error for Euler's method (Equation (9)). The first term in this series, $(-h^2/2)y''_r$, is the most important because it contains the lowest power of h. For sufficiently small values of h all the other terms become relatively small compared with this first term. For this reason we give the first term a special name and call it the **principal term in the local truncation error**.

When comparing two methods it is the principal terms in their local truncation errors which will normally be compared and in particular the power of h in the principal term. The method with the higher power of h in its principal term will give a smaller local truncation error if h is sufficiently small.

I must emphasize that the local truncation error does not, in itself, give us any idea of the *accumulated* error after n steps. We shall deduce some results for accumulated errors in Section 3. The local truncation error does give us some practical information concerning the choice of step-size. Suppose we halve the step-size, h, in Euler's method. Since the principal term involves h^2, halving the step-size reduces the local truncation error by a factor of 4, approximately. Remember, however, that halving the step-size means that we have to do twice as many computations to determine the approximation at a given value of x, so that there is more accumulation of errors.

In the following example we shall see how the accumulated error behaves for two different values of h.

Example 2

Apply Euler's method to the differential equation

$$y' = x + y \qquad \text{with } y(0) = 0,$$

using step-sizes $h = 0.1$ and $h = 0.05$ to compute the values of Y_r at $x = 0.1$, 0.2 and 0.3.

Solution

$h = 0.1$

r	x_r	Y_r
0	0	0
1	0.1	0.00
2	0.2	0.01
3	0.3	0.031

$h = 0.05$

r	x_r	Y_r
0	0	0
1	0.05	0.0000
2	0.1	0.0025
3	0.15	0.00763
4	0.2	0.01551
5	0.25	0.02628
6	0.3	0.04010

The true solution at $x = 0.3$ is $y = 0.0499$. The graph in Figure 5 opposite indicates the growth of errors.

Exercise 3

The Taylor series method of order 2 is given by

$$Y_{r+1} = Y_r + hY'_r + \frac{h^2}{2}Y''_r.$$

Calculate the local truncation error for this method. Write down the principal term in the local truncation error.

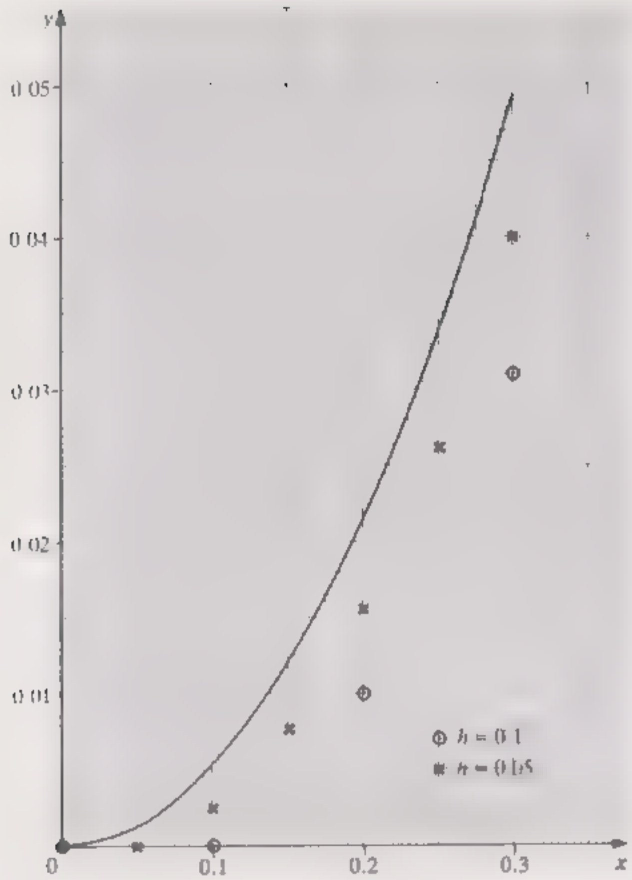


Figure 5. Graph comparing the numerical results for two values of h in Example 2.

Although halving the step-size reduces the error per step by a quarter (approximately), the error in the value of Y at $x = 0.3$ has only been reduced by half (approximately). We shall investigate this further in Section 3.

Exercise 4

Calculate the local truncation error for the Taylor series method of order 3. Write down the principal term in this error. By what factor would you expect the local truncation error to decrease if the step-size were halved?

Exercise 5

Calculate the local truncation error for the Taylor series method of order n.

(The result of this exercise is important and is repeated in the summary.)

Exercise 6

The Taylor series method of order 2 was applied to the differential equation

$$y' = x + y \qquad \text{with } y(0) = 0.$$

Two sets of results were obtained for step sizes $h = 0.1$ and $h = 0.05$ for values of x up to $x = 0.3$

$h = 0.1$

r	x_r	Y_r
0	0	0
1	0.1	0.005
2	0.2	0.021025
3	0.3	0.049232

$h = 0.05$

r	x_r	Y_r
0	0	0
1	0.05	0.00125
2	0.1	0.005127
3	0.15	0.011764
4	0.2	0.021305
5	0.25	0.033897
6	0.3	0.049696

The true values are $y = 0.005171$ at $x = 0.1$
$y = 0.021403$ at $x = 0.2$
$y = 0.049859$ at $x = 0.3$

Compute the errors at $x = 0.1$, 0.2 and 0.3 for the two step-sizes. By approximately what factor have the errors been reduced by halving the step-size?

[*Solutions on p. 51*]

1.3 Local truncation errors for implicit methods

Local truncation errors for implicit methods, such as the trapezoidal method, cannot be calculated using the techniques you have met in the last subsection. In the trapezoidal method we have

$$Y_{r+1} = Y_r + \frac{h}{2}(Y'_r + Y'_{r+1}). \tag{10}$$

If we assume that Y_r lies on a solution curve we have

$$Y_{r+1} = y_r + \frac{h}{2}(y'_r + Y'_{r+1}) \tag{11}$$

and the difficulty is that we do not know enough about Y'_{r+1} to be able to compare this formula for Y_{r+1} with the Taylor series expansion for y_{r+1}. However we *can* compute the *principal term* in the local truncation error by replacing Y'_{r+1} by y'_{r+1} in Equation (11). Although the error in Y'_{r+1} affects the other terms in the local truncation error, it does *not* affect the first term since Y'_{r+1} is multiplied by h in Equation (11). In effect we are saying that the principal term in the local truncation error can be computed by assuming that, in Equation (10), all the right-hand side values can be replaced by their true values.

The Taylor series expansion for y'_{r+1} is

$$y'_{r+1} = y'_r + hy''_r + \frac{h^2}{2}y'''_r + \cdots .$$

Substituting this in the approximate expression for Y_{r+1} gives

$$\begin{aligned} Y_{r+1} &\simeq y_r + \frac{h}{2}(y'_r + y'_{r+1}) \\ &= y_r + \frac{h}{2}y'_r + \frac{h}{2}(y'_r + hy''_r + \frac{h^2}{2}y'''_r + \cdots). \end{aligned} \tag{12}$$

Now the true value for y_{r+1} is given by the Taylor series at x_r as

$$y_{r+1} = y_r + hy'_r + \frac{h^2}{2}y''_r + \frac{h^3}{6}y'''_r + \cdots . \tag{13}$$

Subtracting Equation (13) from Equation (12) gives

$$Y_{r+1} - y_{r+1} \simeq y_r + \frac{h}{2}y'_r + \frac{h}{2}\{y'_r + hy''_r + \frac{h^2}{2}y'''_r \cdots\} - y_r - hy'_r - \frac{h^2}{2}y''_r - \frac{h^3}{6}y'''_r - \cdot$$

$$\frac{h^3}{12}y'''_r +$$

Thus the principal term in the local truncation error for the trapezoidal method is $(h^3\; 12)y'''_r$.

The following procedure summarizes the technique for computing the principal term in the local truncation error and can be used for both implicit and explicit methods For an explicit method you get all the terms in the local truncation error whereas for an implicit method you can only get the principal term.

> **To find the principal term in the local truncation error**
>
> 1. Write down the recurrence relation for the method as a formula for Y_{r+1}.
> 2. Replace Y by y in all symbols on the right-hand side of this recurrence relation.
> 3. Expand this new recurrence relation as a power series in h using Taylor series about x_r. This gives Y_{r+1} in terms of y_r, y'_r, …
> 4. By subtracting the Taylor series
>
> $$y_{r+1} = y_r + hy'_r + \frac{h^2}{2}y''_r + \cdots$$
>
> express $Y_{r+1} - y_{r+1}$ in powers of h.
> 5. The term in the lowest power of h is the principal term in the local truncation error.

We can compare, for example, the trapezoidal method with Euler's method. Since the principal terms in the local truncation error involve h^3 and h^2 respectively we would expect much better results from the trapezoidal method for sufficiently small values of h.

Exercise 7

Consider a method which uses the recurrence relation

$$Y_{r+1} = Y_r + hY'_{r+1}$$

Determine the principal term in the local truncation error for this implicit method. What other method have you seen whose principal term involves the same power of h?

Exercise 8

(i) For the trapezoidal method, if you halve the step-size what is the approximate factor by which the error per step reduces? Which Taylor series method would reduce the error per step by the same factor?

(ii) Using the results of Exercise 6, by what factor would you expect the *accumulated* errors to reduce if the step-size were halved for the trapezoidal method?

Exercise 9

The trapezoidal method was applied to the differential equation

$$y' = x + y \qquad \text{with } y(0) = 0.$$

Two step-sizes, $h = 0.1$ and $h = 0.05$, were used as in Exercise 6. The following tables of results were obtained

$h = 0.1$

r	x_r	Y_r
0	0	0
1	0.1	0.005263
2	0.2	0.021607
3	0.3	0.050197

$h = 0.05$

r	x_r	Y_r
0	0	0
1	0.05	0.001282
2	0.1	0.005194
3	0.15	0.011871
4	0.2	0.021454
5	0.25	0.034092
6	0.3	0.049943

Comparing these results with the true solution at $x = 0.1$, 0.2 and 0.3 we have the following table of errors

x_r	$h = 0.1$	$h = 0.05$
0.1	0.000092	0.000023
0.2	0.000204	0.000051
0.3	0.000338	0.000084

By what factor (approximately) have the accumulated errors been reduced by halving the step-size?
[*Solutions on pp. 51 52*]

Summary of Section 1

1. **Integration methods** for the numerical solution of differential equations are derived from numerical integration techniques.
2. The **trapezoidal method** is an integration method and gives rise to the recurrence relation
$$Y_{r+1} = Y_r + \frac{h}{2}(Y'_r + Y'_{r+1}).$$
3. The trapezoidal method can be applied to the linear differential equation
$$y' = l(x)y + k(x)$$
using the procedure on page 8.
4. The trapezoidal method is an example of an implicit method. An **implicit** method for solving a differential equation gives rise to a recurrence relation involving Y'_{r+1} or higher derivatives at x_{r+1}. A method is **explicit** if the recurrence relation does not involve Y'_{r+1} or higher derivatives at x_{r+1}. Implicit methods are difficult to use with non-linear equations.
5. The **local truncation error** for a numerical method for solving differential equations is the error induced in Y_{r+1} assuming that Y_r lies on some solution curve, y, so that $Y_r = y_r$.
6. The **principal term in the local truncation error** is the term with the lowest power of h. The principal term can be determined using the procedure on page 13.
7. The principal terms in the local truncation error for various methods are as follows:

method	principal term
(i) Euler's method	$-\frac{h^2}{2}y''_r$
(ii) Taylor series method of order 2	$-\frac{h^3}{6}y'''_r$
(iii) Taylor series method of order n	$-\frac{h^{n+1}}{(n+1)!}y^{(n+1)}_r$
(iv) Trapezoidal method	$\frac{h^3}{12}y'''_r$

2 The predictor-corrector method (Television Section)

2.1 Pre-broadcast notes

The television programme for this unit investigates ways of obtaining numerical solutions to the logistic equation
$$y' = 10y\left(1 - \frac{y}{1000}\right) \qquad \text{with } y(0) = 100.$$

You met this equation in *Unit 3* Section 2.

This equation models the growth in a population where the initial population is 100 and the population stops growing if it reaches 1000.

This equation has an analytic solution which you will be asked to find in Exercise 1, so you may be wondering why we should want to determine numerical approximations for this problem. There are several reasons:

(i) We can compare our numerical solution with the true solution.

(ii) If we have to obtain numerical approximations to a closely related differential equation which *cannot* be solved analytically, we may be able to use information we derived solving this one.

(iii) The non-linearity in this equation will highlight some of the difficulties in using numerical methods and how these difficulties can be overcome.

The television programme is divided into three parts:

(i) The integration methods developed in Subsection 1.1 will be reviewed and the methods applied to the logistic equation. An important feature of the programme is the way in which the graphs of y' and y inter-relate.

(ii) The failure of the trapezoidal method gives rise to a method based on using Euler's method and the trapezoidal method together in a method known as a predictor-corrector method. We will look at this method in more detail after the programme.

(iii) The final part of the programme looks at the importance of choosing a small enough step-size, h, so that the numerical solution is qualitatively the same as the true solution. This topic will be explored more fully in Section 3.

Exercise 1

Use the method of separation of variables to solve the differential equation

$$y' = 10y\left(1 - \frac{y}{1000}\right).$$

What is the particular solution if $y(0) = 100$?

Sketch a graph of the particular solution for $0 \le x \le 1$. (The easiest way to do this is to plot a few points.)

[*Solution on p. 52*]

Now watch the television programme 'Integrating by numbers'.

TV

TV19

2.2 Programme notes

Euler's method, as an integration method, can be considered as a means of building up a graph of the approximate solution, y, by calculating the approximate area under the graph of y', as described in Subsection 1.1. In Subsection 1.1 we derived the basic equation for an integration method as

$$y_1 = y_0 + \int_{x_0}^{x_1} y'\,dx \qquad \text{(from Equation (3) of Subsection 1.1)}$$

where $y' = m(x, y)$ is the differential equation to be solved.

Euler's method was derived by approximating the integral by the area of a rectangle of width $h = x_1 - x_0$ and height y'_0 (see Figure 1(a)),

i.e. $$\int_{x_0}^{x_1} y'\,dx \simeq hy'_0.$$

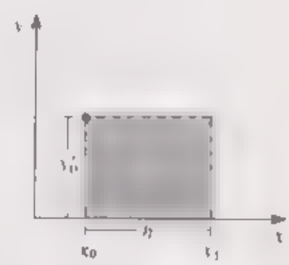


Figure 1(a). Graph of y' against x showing the area for Euler's method.

The value of y'_0 can be found by substituting the initial condition y_0 in the differential equation to give

$$y'_0 = m(x_0, y_0).$$

The value of the area, hy'_0 is added onto the value of y_0 to give an approximation for y_1 so that

$$Y_1 = y_0 + hy'_0$$

as shown in Figure 1(b)

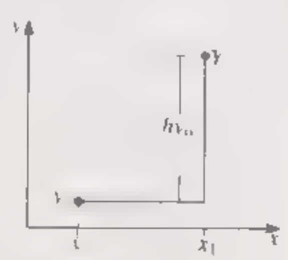


Figure 1(b). How y_0 is incremented to obtain Y_1

Once we have found Y_1 we can repeat the process to find Y'_1, then Y_2 and so on. In this way we can build up the graphs of both y and y'. Summarizing the method we have a three-stage procedure:

(i) Evaluate Y'_r as $Y'_r = m(x_r, Y_r)$ and plot this point on the y' graph

(ii) Calculate the area of the rectangle height Y'_r and width $h = x_{r+1} - x_r$.

(iii) Increment Y_r by the value of this area to obtain Y_{r+1}.

The important thing here is to see how the graphs of y and y' inter-relate.

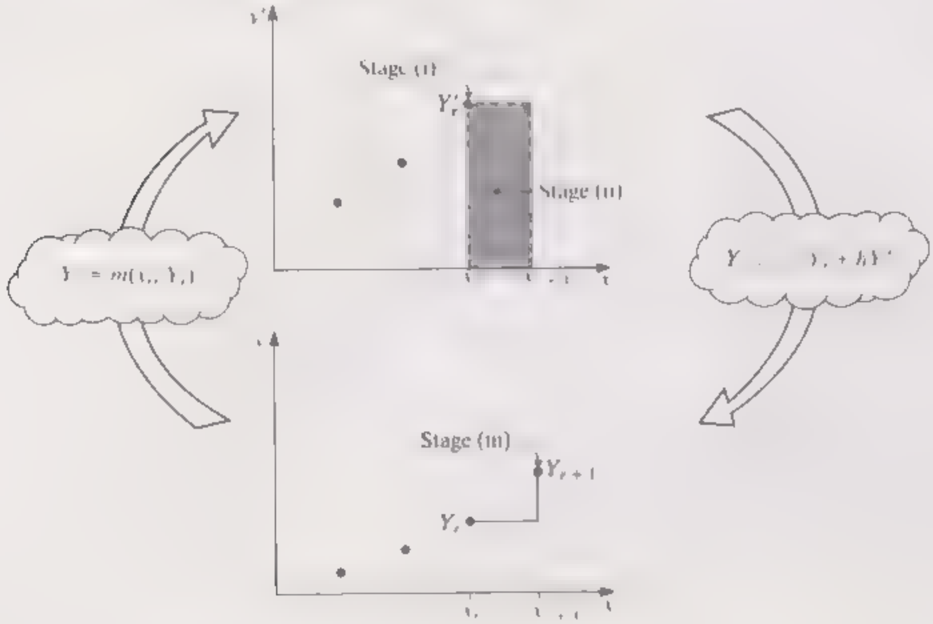


Figure 2. Schematic diagram for Euler's method.

If we try to get the same geometrical insight into the trapezoidal method we come unstuck because the method requires both y'_0 and Y'_1 in order to be able to calculate Y_1 using the formula

$$Y_1 = y_0 + \frac{h}{2}(y'_0 + Y'_1).$$

To find Y_1 we must increment y_0 by the area of a trapezium and we cannot find the area of the required trapezium geometrically because the only information we have is the values of y_0 and $y'_0 = m(x_0, y_0)$.

For linear equations we can do some algebraic manipulation to get round this difficulty, as described in Subsection 1.1. For the logistic equation we could do some algebraic manipulation and determine Y_1 by solving a quadratic equation. This process is much more involved than Euler's method and in the programme this method is not pursued.

The second part of the programme indicates how we can get round the difficulty in using the trapezoidal method for non-linear differential equations by using a combination of Euler's method and the trapezoidal method. To apply the trapezoidal method we need Y'_1 in order to compute Y_1 as

$$Y_1 = y_0 + \frac{h}{2}(y'_0 + Y'_1)$$

and Y'_1 depends on Y_1. The way round this difficulty is to use Euler's method to obtain a crude approximation for Y'_1 which we can then use in the trapezoidal formula. Because this crude approximation is multiplied by h in the formula we can still get quite a good approximation, Y_1.

Example 1

To see how the method works I will compute a value for Y_1 at $x = 0.1$ for the logistic equation using this new method.

Solution

The method is in four stages:

(i) We know that $Y_0 = 100$, so I can evaluate Y_0' using the differential equation as $Y_0' = 10Y_0\left(1 - \dfrac{Y_0}{1000}\right)$ giving

$$Y_0' = 10 \times 100\left(1 - \frac{100}{1000}\right) = 900.$$

$Y_0' = m(x_0, Y_0)$

Y_0' enables us to calculate the area of the rectangle in Figure 3(a).

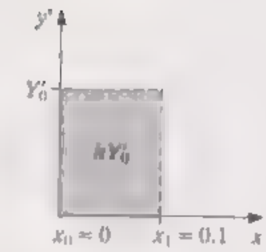


Figure 3(a)

(ii) Using Euler's method I get a crude approximation for Y_1 in Figure 3(b) as

$$Y_1^* = Y_0 + hY_0' = 100 + 0.1 \times 900$$
$$= 190.$$

I've labelled this with an asterisk to indicate that it is a crude approximation. We are going to do better.

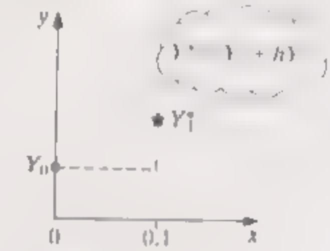


Figure 3(b)

(iii) This is not a particularly good approximation. However I can use it to obtain a value for the derivative $Y_1^{*\prime}$, using the logistic equation, as

$Y_1^{*\prime} = m(x_1, Y_1^*)$

$$Y_1^{*\prime} = 10Y_1^*\left(1 - \frac{Y_1^*}{1000}\right) = 10 \times 190\left(1 - \frac{190}{1000}\right)$$
$$= 1539.$$

This is shown in Figure 3(c).

Figure 3(c)

(iv) Look at Figure 3(d). Since we now have values for Y_0' and $Y_1^{*\prime}$ we can calculate the area of the trapezium. Adding it to Y_0 we obtain an improved approximation Y_1 as

$$Y_1 = Y_0 + \frac{h}{2}\left(Y_0' + Y_1^{*\prime}\right)$$
$$= 100 + \frac{0.1}{2}(900 + 1539)$$
$$= 221.95,$$

as shown in Figure 3(e).

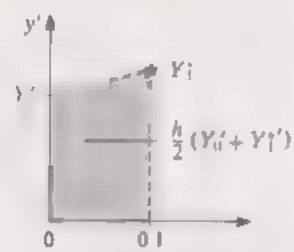


Figure 3(d)

This is much closer than Y_1^* to the true value of y at $x = 0.1$, which is 231.97 correct to 2 decimal places.

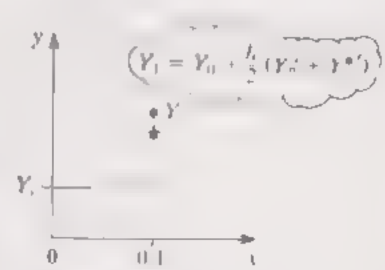


Figure 3(e)

The method outlined above is an example of a **predictor-corrector method**. In Example 1 Euler's method was used to predict a value for Y_1. This value for Y_1 was then corrected using the trapezoidal method.

Predictor-corrector methods provide one of the most widely used techniques for solving differential equations. They are based on the idea of using an explicit method to obtain a prediction and then using a more accurate implicit method to correct this prediction.

The four stages of the procedure I used in Example 1 can be summarized as:

(i) *Evaluate* Y_0' using the differential equation.

(ii) *Predict* Y_1^* using Euler's method.

(iii) *Evaluate* $Y_1^{*\prime}$ using the differential equation.

(iv) *Correct* Y_1 using the trapezoidal method

Here is the same procedure in diagramatic form.

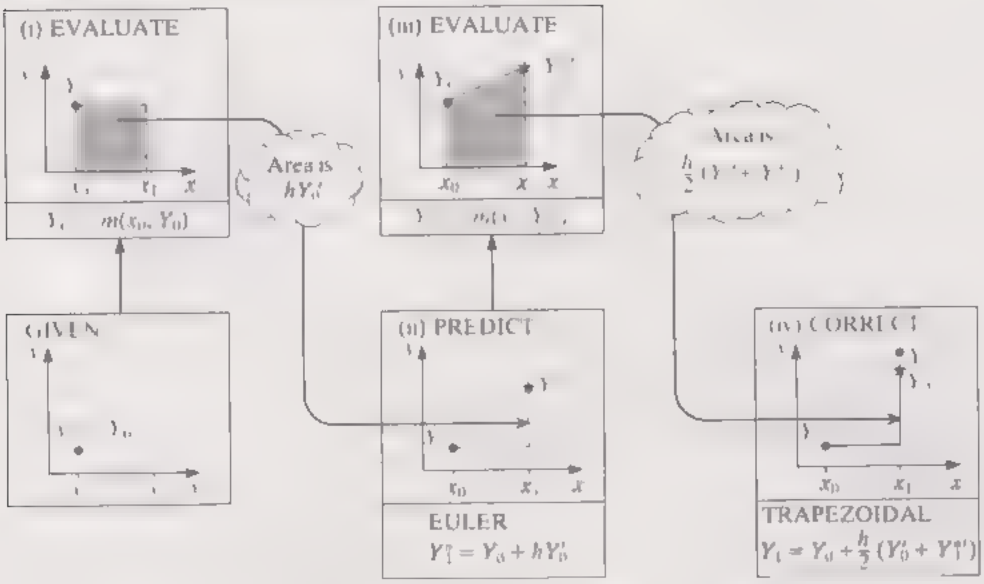


Figure 4. Schematic diagram of the Euler-trapezoidal predictor-corrector method.

Example 2

When the procedure above was used to compute Y_1 for the logistic equation, the value $Y_1 = 221.95$ was obtained. I can now use this value as the starting point in the computation of Y_2.

Solution

We can use the four stages again as follows:

(i) Evaluate $Y_1' = 10Y_1\left(1 - \frac{Y_1}{1000}\right) = 10 \times 221.95\left(1 - \frac{221.95}{1000}\right)$

$= 1726.882.$

(ii) Predict Y_2^* using Euler's method as

$$Y_2^* = Y_1 + hY_1'$$
$$= 221.95 + 0.1 \times 1726.882$$
$$= 394.6382.$$

hY_1' is the area of the rectangle.

(iii) Evaluate $Y_2^{*\prime} = 10Y_2^*\left(1 - \frac{Y_2^*}{1000}\right) = 10 \times 394.6382\left(1 - \frac{394.6382}{1000}\right)$

$= 2388.9889.$

(iv) Correct Y_2 using the trapezoidal method.

$$Y_2 = Y_1 + \frac{h}{2}(Y_1' + Y_2^{*\prime})$$
$$= 221.95 + 0.05(1726.882 + 2388.9889)$$
$$= 427.74355.$$

$\frac{h}{2}(Y_1' + Y_2^{*\prime})$ is the area of the trapezium

Exercise 2

Follow the method of Example 2 to compute values for Y_3 and Y_4 for the logistic equation.

[*Solution on p. 52*]

Thus we can use the Euler-trapezoidal predictor-corrector method to compute Y_1, $Y_2, \ldots, Y_n$, as summarized in the following procedure.

The predictor-corrector method (Euler-trapezoidal)

1. You are given a differential equation

$$y' = m(x, y)$$

and a value for $y(x_0)$.

2. To apply the predictor-corrector method, choose a step-size h, and calculate Y_1, Y_2, $Y_3, \ldots$ by repeating the following cycle of steps starting with $Y_0 = y(x_0)$:

   (i) Evaluate $Y'_r = m(x_r, Y_r)$.

   (ii) Predict using Euler's method:

   $$Y^*_{r+1} = Y_r + hY'_r.$$

   (iii) Evaluate $Y^{*\prime}_{r+1} = m(x_r, Y^*_{r+1})$.

   (iv) Correct using the trapezoidal method:

   $$Y_{r+1} = Y_r + \frac{h}{2}(Y'_r + Y^{*\prime}_{r+1}).$$

3. Y_r is an approximation to y_r.

Note that any predictor-corrector method is an explicit method. (You will be asked to show this for the Euler-trapezoidal method in Exercise 5.)

In general, predictor-corrector methods use an explicit method to predict Y^*_{r+1} and a more accurate implicit method to correct Y_{r+1}.

Although the derivation of the local truncation error for a predictor-corrector method is beyond the scope of this course, it can be shown that the principal term in the local truncation error for the Euler-trapezoidal method involves h^3 (as does the principal term in the local truncation error for the trapezoidal method).

The following table gives the computed values for the logistic equation for each of the three methods discussed in the programme and also the true solution, obtained using the formula from Exercise 1 The results for the trapezoidal method were obtained by solving a quadratic equation at each step.

Solutions to the logistic equation, with $h = 0.1$.

x_r	Euler	trapezoidal	predictor-corrector	true solution
0	100	100	100	100
0.1	190	235	222	232
0.2	344	448	428	451
0.3	570	681	660	691
0.4	815	852	823	858
0.5	966	942	911	943
0.6	999	980	956	978
0.7	1000	993	978	992
0.8	1000	998	989	997
0.9	1000	999	994	999
1.0	1000	1000	997	1000

I have plotted these points on a graph shown in Figure 5. Note that the trapezoidal method gives the best results with the predictor-corrector method initially giving much better results than Euler's method.

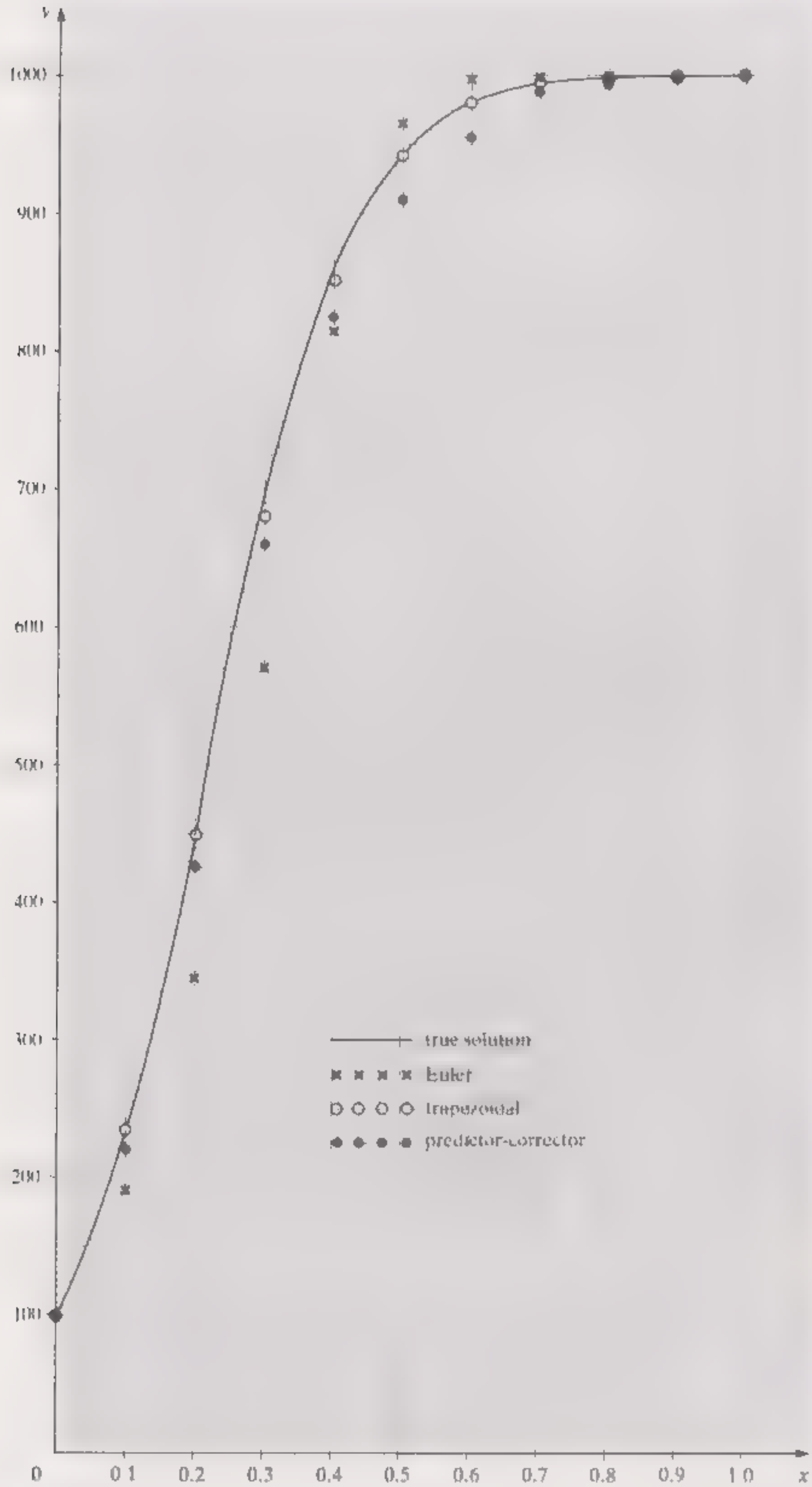


Figure 5. Comparison of results for the logistic equation.

In the final part of the programme we look at what happens if we use different step-sizes, h The first method we look at is Euler's method. The four graphs in Figure 6 show the results for $h = 0.1$, 0 15, 0.2 and 0.25 The logistic curve is also shown for comparison.

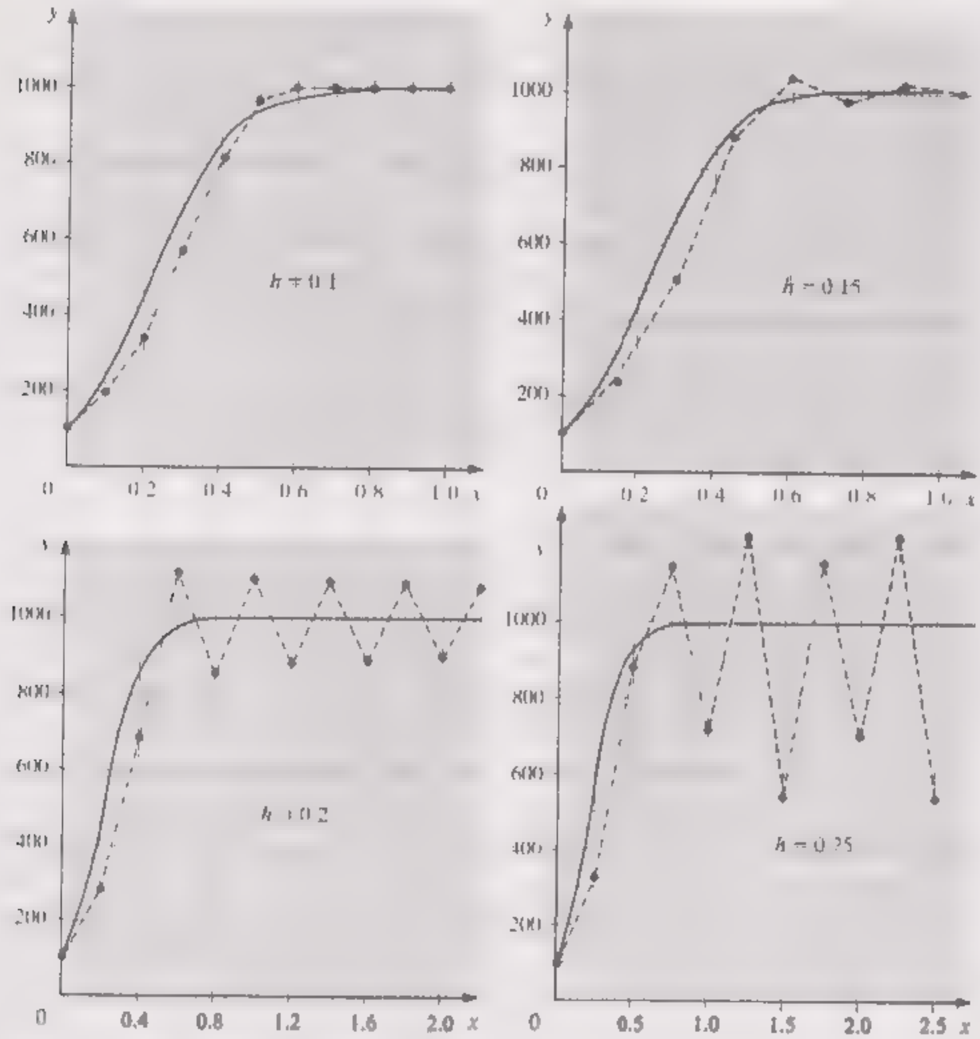


Figure 6. Euler's method for different step-sizes.

The results show that, for $h = 0.1$ and 0 15, the approximate solution at least has approximately the same qualitative behaviour as the analytical solution. The saw-tooth behaviour near $y = 1000$ of the solution for $h = 0.2$ and 0.25 is clearly undesirable. This will be discussed in Section 3 but the graphs do indicate that, for some problems at least, we have to be careful about our choice of step-size if we want to get reasonable results.

The trapezoidal method, on the other hand, behaves well for this problem as the graphs in Figure 7, with step-sizes $h = 0.1$, 0.2, 0.5 and 1.0, illustrate.

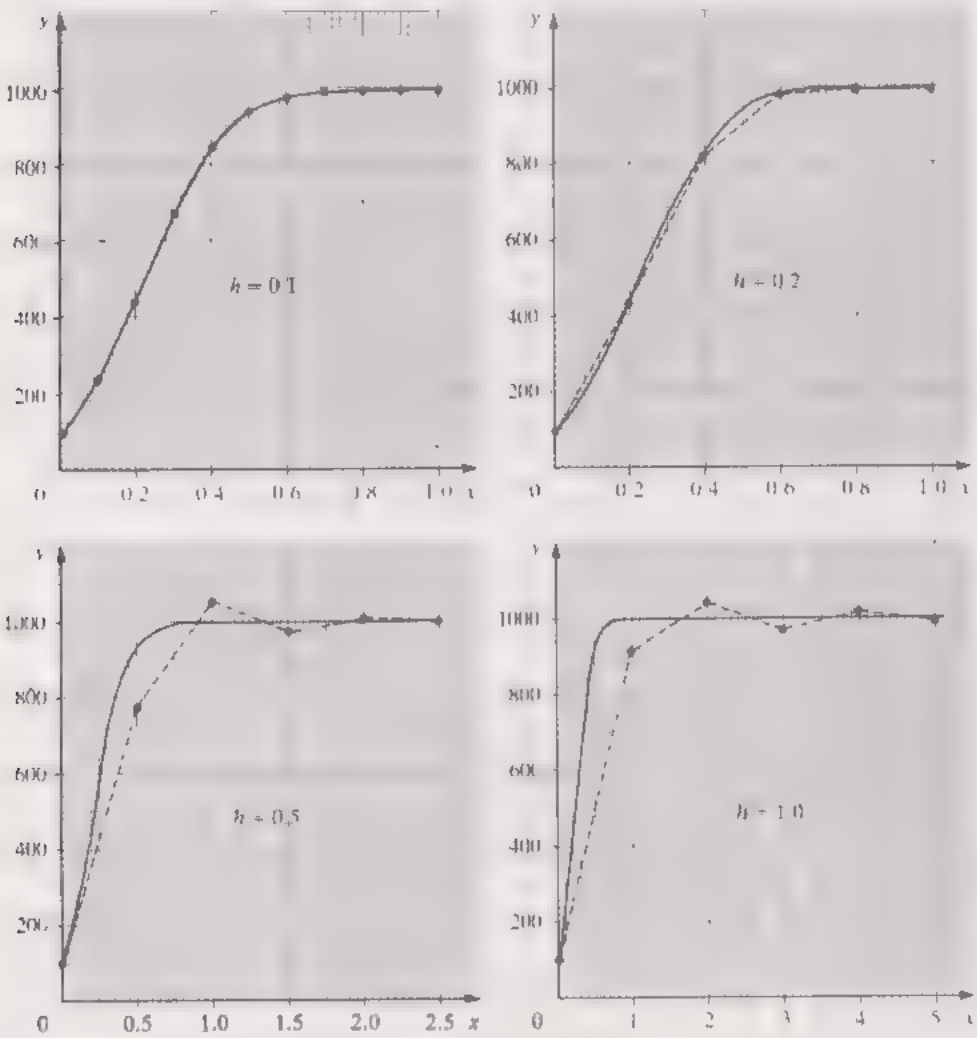


Figure 7. The trapezoidal method for different step-sizes.

While numerical results for $h = 0.5$ and 1.0 are not very good they do converge to the limit $y = 1000$ as x increases and reasonably approximate the shape of the logistic curve.

So what about the predictor-corrector method? It is a combination of the Euler and trapezoidal methods. Unfortunately, as indicated in the graphs in Figure 8, this method also behaves badly for larger values of h.

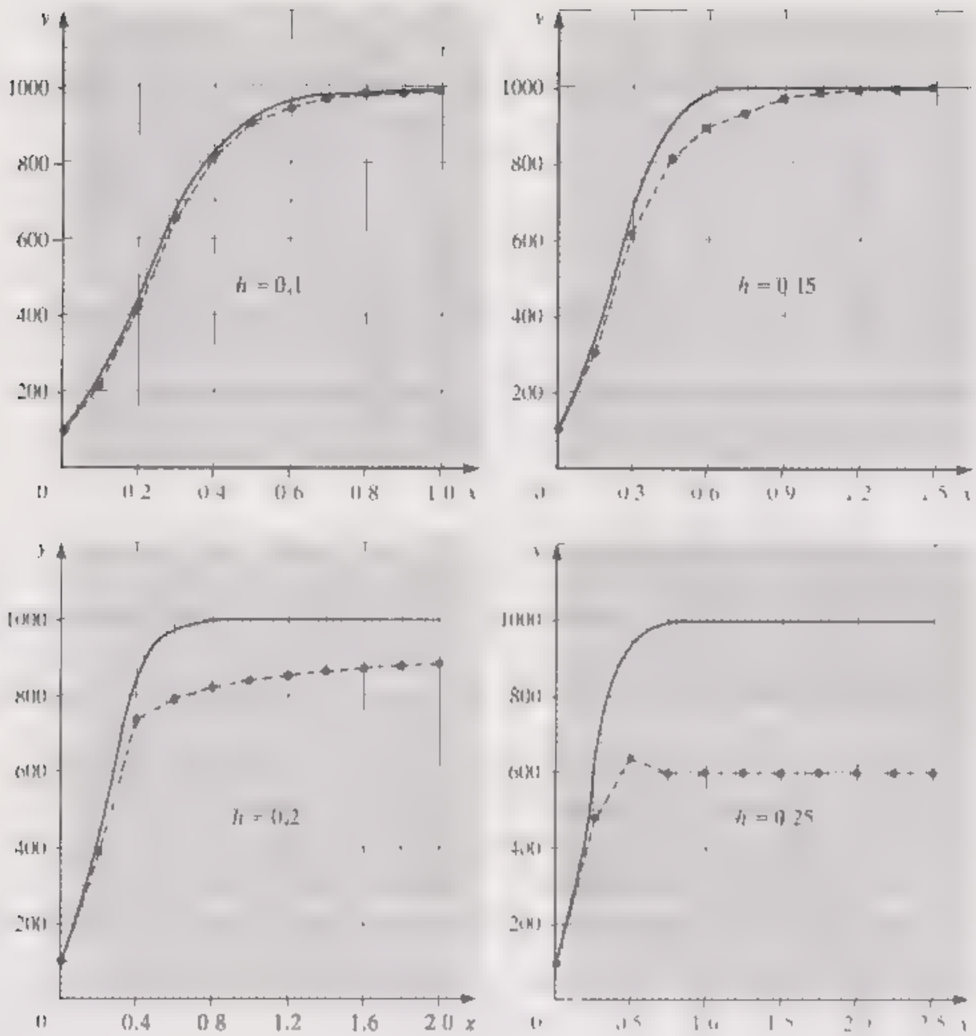


Figure 8. The predictor-corrector method for different step-sizes.

The numerical results follow the shape of the logistic curve for $h = 0.1$ and $h = 0.15$, but for $h = 0.2$ the numerical approximations are converging extremely slowly towards the limit value of 1000 while for $h = 0.25$ the numerical results converge rapidly to the *wrong* limit.

Exercise 3

Use the predictor-corrector method (Euler-trapezoidal) with $h = 0.1$ to determine approximate solutions for y at $x = 0.1$ and $x = 0.2$ for the differential equation

$$y' = \log y + x \qquad \text{with } y(0) = 1.$$

Quote your results to 3 decimal places.

Exercise 4

Use the predictor-corrector method (Euler-trapezoidal) with $h = 0.2$ to determine approximate solutions for y at $x = 0.2$ and $x = 0.4$ for the differential equation

$$y' = \sin y \qquad \text{with } y(0) = 1.$$

Quote your results to 3 decimal places.

Exercise 5

Eliminate Y^*_{r+1} and $Y^{*\prime}_{r+1}$ from the recurrence relations for the predictor-corrector method

$$Y^*_{r+1} = Y_r + hY'_r \qquad \text{where } Y'_r = m(x_r, Y_r)$$

and $$Y_{r+1} = Y_r + \frac{h}{2}(Y'_r + Y^{*\prime}_{r+1}) \qquad \text{where } Y^{*\prime}_{r+1} = m(x_{r+1}, Y^*_{r+1})$$

and hence verify that this method is an explicit method

[*Solutions on pp. 52–53*]

Summary of Section 2

1 The television programme looks at several methods of obtaining numerical solutions to the logistic equation

$$y' = 10y\left(1 - \frac{y}{1000}\right) \qquad \text{with } y(0) = 100$$

including Euler's method and the trapezoidal method.

2. A **predictor-corrector method** uses an explicit method to obtain a crude approximation to the solution of a differential equation at x_{r+1} and then uses a more accurate implicit method to obtain an improved approximation at x_{r+1}.

3. The predictor-corrector method based on Euler's method and the trapezoidal method overcomes the difficulty in using the trapezoidal method to obtain numerical solutions for non-linear differential equations. The procedure for applying this method is given on page 19. The principal term in the local truncation error for this method involves h^3.

4. For Euler's method and the predictor-corrector method we need to choose the step-size, h, carefully in order to obtain reasonable results.

3 The analysis of numerical methods

From the empirical results of Sections 1 and 2 we have seen that

(i) it is possible to approximate differential equations using a variety of different methods, each of which leads to a recurrence relation;

(ii) the smaller the step-size, h, the better the approximation;

(iii) larger values of h may give nonsense.

In this section we want to look at properties of methods theoretically so that you can evaluate any new method you might meet The analysis will also help you to choose the step-size required to give reasonable results.

We shall introduce three new concepts:

(1) *consistency*, which is related to (i) above,
(2) *convergence*, which is related to (ii) above,
(3) *stability*, which is related to (iii) above.

For simplicity we are only going to consider **one-step methods** in which the differential equation is approximated by a first-order recurrence relation All the methods you have met so far have been one-step methods, but in Section 4 you will see a two-step method in which the recurrence relation is of second order (i.e. the formula for Y_{r+1} involves Y_{r-1} as well as Y_r). The analysis can be extended to multi-step methods but we do not have time to do so in this unit.

3.1 Consistency

One of the fundamental requirements of any numerical method that we use to find approximate solutions to a differential equation is that the recurrence relation to be used should be a reasonable approximation to the differential equation itself In other words the recurrence relation must be *consistent* with the differential equation. If this requirement is not satisfied we cannot hope that the solution obtained using the recurrence relation is a reasonable approximation to the solution of the differential equation.

As in the definition of the local truncation error, consistency is concerned with the *local* behaviour of the method. We assume that Y_r lies on some solution curve so that $Y_r = y_r$. For implicit methods we also assume that any other values on the right-hand side of the recurrence relation are correct.

Any one-step method, including the predictor-corrector method of Section 2, must involve a recurrence relation of the form

$$Y_{r+1} = Y_r + h\phi(x_r, Y_r, Y_{r+1}, h) \tag{1}$$

where the function ϕ depends on the method used. For example the Taylor series method of order 2 uses the recurrence relation

$$\begin{aligned} Y_{r+1} &= Y_r + hY'_r + \frac{h^2}{2}Y''_r \\ &= Y_r + h\left(Y'_r + \frac{h}{2}Y''_r\right) \end{aligned}$$

and so, for this method, we have

$$\phi(x_r, Y_r, Y_{r+1}, h) = Y'_r + \frac{h}{2}Y''_r.$$

Using the condition that $Y_r = y_r$ and that the other right-hand side values are correct in Equation (1), we have

$$Y_{r+1} = y_r + h\phi(x_r, y_r, y_{r+1}, h).$$

Rearranging this equation gives

$$\frac{Y_{r+1} - y_r}{h} = \phi(x_r, y_r, y_{r+1}, h). \tag{2}$$

The left-hand side of Equation (2) represents the slope of the line joining the points (x_r, y_r) and (x_{r+1}, Y_{r+1}). As h gets smaller and smaller we require that Equation (2) should be a better and better approximation to the differential equation

$$y' = m(x, y) \tag{3}$$

at x_r. This condition holds when $h = 0$ provided that

$$\phi(x_r, y_r, y_r, 0) = m(x_r, y_r).$$

(Since $y_{r+1} = y(x_r + h)$ we have $y_{r+1} = y_r$ when $h = 0$.) We thus have a fairly simple test for consistency.

> A one-step method is said to be **consistent** with the differential equation (3) if the recurrence relation can be expressed in the form of Equation (1) with
>
> $$\phi(x_r, y_r, y_r, 0) = m(x_r, y_r).$$

The definition of consistency can alternatively be stated as

$$\lim_{h \to 0} \frac{Y_{r+1} - y_r}{h} = m(x_r, y_r)$$

Example 1

For the Taylor series method of order 2 we have

$$\phi(x_r, y_r, y_{r+1}, h) = y'_r + \frac{h}{2}y''_r$$

so that putting $h = 0$ in this equation gives

$$\begin{aligned} \phi(x_r, y_r, y_r, 0) &= y'_r \\ &= m(x_r, y_r) \end{aligned}$$

as required.

Hence the method is consistent with the differential equation (3).

Exercise 1

Which of the following methods are consistent with the differential equation

$$y' = m(x,y)\ ?$$

(i) $Y_{r+1} = Y_r + \frac{h}{3}(Y'_r + 2Y'_{r+1})$

(ii) $Y_{r+1} = Y_r + \frac{h}{4}(3Y'_r - Y'_{r+1})$.

Exercise 2

What condition must be imposed on the values of a and b for the method given by

$$Y_{r+1} = Y_r + h(aY'_r + bY'_{r+1})$$

to be consistent?

[*Solutions on p. 53*]

3.2 Convergence of one-step methods

In Exercises 6 and 9 of Section 1 you were asked to compare the results of using a method with two different step-sizes. In Exercise 6 you saw that halving the step-size for the Taylor series method of order 2 reduced the errors by a factor of approximately 4. In Exercise 9 you obtained a similar result for the trapezoidal method.

What would have happened if we had halved the step-size again? Well, commonsense tells us we could expect a further reduction in the error by another factor of 4 each time. Thus if we continue halving the step-size (and doubling the number of calculations) we would theoretically obtain answers to any accuracy we require. (In practice there is a limit to the accuracy that we can achieve because of the build-up of rounding errors in the calculations.) Suppose we assume that all calculations are carried out exactly. Then our empirical evidence from Exercises 6 and 9 in Section 1 indicates that the solutions *converge* in each case. That is, if we take some particular point x^* (say $x^* = 0.2$) and compute the approximation to y at that point using smaller and smaller step-sizes, then these approximations get closer and closer to the true solution at that point. Figure 1 illustrates how the numerical solution might converge to the true solution.

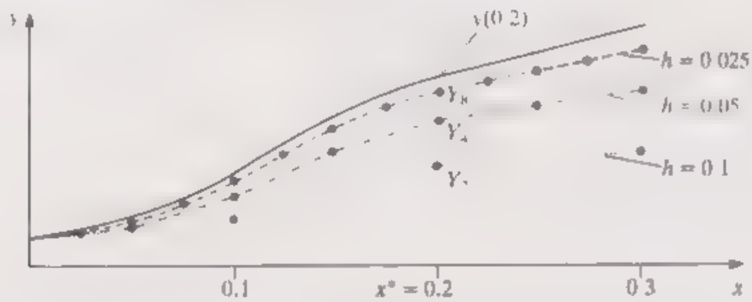


Figure 1. Graph showing how the numerical solution could converge to the solution of the differential equation.

For a given numerical method let us look at the error at a fixed value of x as h changes.

(i) Suppose we take a fixed value of x, say x^*.

(ii) As h decreases so the number of steps, N^*, from x_0 to x^* increases such that

$$x^* = x_0 + N^*h.$$

The **global error** at x^* is the error

$$Y_{N^*} - y(x^*).$$

We are now in a position to define *convergence*.

1. You are given a problem of the form

$$y' = m(x,y) \qquad \text{with } y(x_0) \text{ given.}$$

2. You wish to use a one-step numerical method to obtain approximations for $x_0 \le x \le b$.
3. The method is **convergent** for this problem if, for all points x^* such that $x_0 \le x^* \le b$, we have

$$\lim_{h \to 0} Y_{N^*} = y(x^*), \qquad \text{where } N^*h = x^* - x_0$$

i.e. the global error at x^* tends to zero as h tends to zero.

Convergence is difficult to prove directly, except for very simple differential equations. However we can state a rather simple but very useful theorem for one-step methods which makes any direct proof of convergence unnecessary.

Theorem 1

For a given problem

$$y' = m(x,y)$$

and a value for $y(x_0)$, where m is well-behaved, a one-step method is convergent if and only if it is consistent. Furthermore if the principal term in the local truncation error involves h^{p+1}, for some integer p, then the global error at x^*, for small h, is of the form

$$Y_{N^*} - y(x^*) \simeq Ch^p$$

where C does not depend on h.

Well-behaved here means that we can differentiate m as often as we wish

This is an important theorem because it gives us a way of evaluating our methods. For example, the trapezoidal method has a local truncation error whose principal term involves h^3. Our theorem tells us that the global error is of the form

$$Y_{N^*} - y(x^*) \simeq Ch^2.$$

While this in itself does not help to put a bound on the error (since we do not know C) it indicates that if we were to halve the step-size h then we could expect the error in Y_{N^*} to decrease by a factor of four, even though we've had to do twice as many calculations to compute Y_{N^*}. This is precisely the result we deduced in Exercises 6 and 9 in Section 1.

The above results indicate that we could classify methods in terms of the principal term in their local truncation errors. A preliminary rank ordering of the methods we have met so far, indicating their practical limitations, would be as follows.

Principal term involves			
h^4	*Taylor series method of order 3* Drawbacks. second and third derivatives required		
h^3	*Predictor-corrector (Euler-trapezoidal)* Drawbacks: none	*Taylor series method of order 2* Drawbacks: second derivative required.	*Trapezoidal method* Drawbacks: for linear equations only
h^2	*Euler's method* Drawbacks: none		

This diagram makes the point that if high accuracy is required then the best method is likely to be a Taylor series method provided the derivatives can be obtained. If less accuracy is required or the derivatives are difficult to compute then the predictor-corrector method may be preferred.

Exercise 3

Show that the trapezoidal method is consistent and hence convergent. How does the global error behave at a point x^* for small values of h? If the step-size were halved, by what factor would the global error at x^* be reduced?

Exercise 4

Use the results of Exercise 5 in Section 2 to show that the predictor-corrector method (Euler-trapezoidal) is consistent.

[*Solutions on p. 53*]

3.3 Stability

There is still an outstanding practical issue in the analysis of one-step methods in that we do not yet know, given a particular differential equation, what value for the step-size, h, to use. The convergence theorem has only told us that, as h tends to zero, the numerical solution tends to the true solution—and only then if we carry out our computations accurately. For a chosen step-size we need to know whether we can expect reasonable results.

In the television programme we saw examples of how numerical solutions can go haywire if the chosen step-size is too large. In each case problems arose because errors, incurred in approximating the differential equation, swamped the solution. In this subsection we are going to investigate a fairly simple test equation to see when these difficulties arise and in the next subsection we will apply our results to more practical problems. The following exercise illustrates what can happen.

Exercise 5

Use Euler's method with $h = 0.1$ to solve the differential equation

$$y' = -30y \qquad \text{with } y(0) = 1. \tag{1}$$

Compare your results with the true solution, $y = e^{-30x}$.

[*Solution on p. 53*]

In *Unit 1* two types of ill-conditioning were defined absolute ill-conditioning and relative ill-conditioning. In this unit we are only going to discuss absolute ill-conditioning.

First we consider the differential equation itself Some differential equations are absolutely ill-conditioned in the sense that small changes in the data result in much larger changes in the solution. Whatever numerical method we use we cannot hope to avoid the build-up of errors, although we can hope that those errors will not swamp the solution In order to deal with this case thoroughly we would need to consider relative ill-conditioning, which is beyond the scope of this unit. On the other hand there are some differential equations which are well-conditioned and it is these problems which particularly concern us here.

When we apply a numerical method to a well-conditioned problem the recurrence relation problem can be absolutely well- or ill-conditioned. In order to discuss this we need a new definition which applies to numerical *methods* rather than to the problems they are intended to solve.

> A numerical method, applied to a differential equation, is **absolutely unstable** if the resulting recurrence relation problem is absolutely ill-conditioned. The method is **absolutely stable** if the recurrence relation problem is absolutely well-conditioned.

To illustrate these ideas we consider the **test problem** given by

$$y' = \alpha y \qquad \text{with } y(0) = 1$$

where α is a known constant This equation can be solved analytically (using, for example, the separation of variables method) giving the solution

$$y = e^{\alpha x}.$$

Figure 2 shows the solution for various values of α.

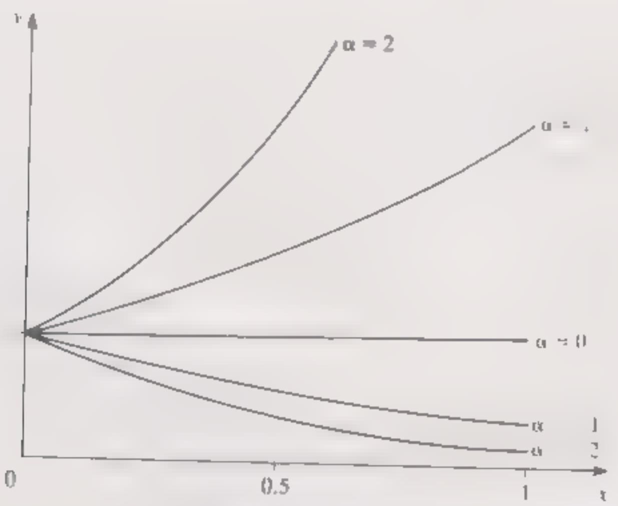


Figure 2. Graphs of $e^{\alpha x}$ for various values of α.

Suppose we perturb the initial condition for this differential equation so that

$$y(0) = 1 + \varepsilon$$

where ε is small. The new solution is

$$y = (1 + \varepsilon)e^{\alpha x} = e^{\alpha x} + \varepsilon e^{\alpha x} = y + \varepsilon e^{\alpha x}.$$

The error in the solution ($\varepsilon e^{\alpha x}$) is significantly larger than the error in the data (ε) whenever $e^{\alpha x}$ is significantly greater than one In this case the differential equation is absolutely ill-conditioned Now for sufficiently large positive x this condition holds whenever α is positive We conclude that the differential equation problem is absolutely ill-conditioned if $\alpha > 0$. By a similar argument the problem is absolutely well-conditioned if $\alpha < 0$.

So what happens if we try to use a numerical method to solve the test problem, Equation (1)? To illustrate the approach we look at Euler's method which uses the recurrence relation

$$Y_{r+1} = Y_r + hY'_r.$$

From the differential equation, $Y'_r = \alpha Y_r$, so we have

$$\begin{aligned} Y_{r+1} &= Y_r + h\alpha Y_r \\ &= (1 + h\alpha)Y_r. \end{aligned}$$

We have a recurrence relation of the form

$$Y_{r+1} = aY_r \qquad (2)$$

where $a = 1 + h\alpha$.

The recurrence relation problem is thus to generate a sequence of numbers $Y_1, Y_2, Y_3 \ldots$ using the recurrence relation (2) with $Y_0 = 1$.

What we now ask is 'when is *this* problem absolutely ill-conditioned?'

From *Unit 1* we know that absolute errors grow whenever

$$|a| > 1.$$

(For example if we put $\bar{Y}_0 = 1 + \varepsilon$ then

$$\bar{Y}_n = (1 + \varepsilon)a^n = a^n + \varepsilon a^n = Y_n + \varepsilon a^n$$

and the absolute error, ε, in Y_0 has induced an error in Y_n of εa^n. This error is larger than the error in the data if $|a| > 1$.)

There are two cases to consider:

Case (i) $\alpha > 0$

Since the step-size h is positive we automatically have

$$a = 1 + h\alpha > 1$$

and the recurrence relation problem is absolutely ill-conditioned. This is not surprising as the original differential equation problem is absolutely ill-conditioned. However, this ill-conditioning may not matter as the growth of the *relative* errors may be tolerable.

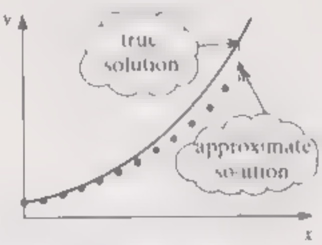


Figure 3. While absolute errors are increasing, relative errors may not increase significantly.

Case (ii) $\alpha < 0$

This is the more interesting case since we know that the differential equation is absolutely well-conditioned.

For negative α we have

$$a = 1 + h\alpha < 1.$$

However this does not imply that the method is absolutely stable. For example, suppose that h and α are such that $h\alpha = -3$. Then

$$a = 1 + h\alpha = 1 - 3 = -2$$

and, since $|a| > 1$, the recurrence relation problem is absolutely ill-conditioned.

In general the condition for absolute ill-conditioning is that the modulus

$$|1 + h\alpha| > 1$$

so that whenever $1 + h\alpha < -1$ we have an absolutely ill-conditioned recurrence relation problem. The condition for absolute ill-conditioning is that

$$h\alpha < -2.$$

Here we have an example in which the original problem is well-conditioned but the numerical method gives rise to a problem which is absolutely ill-conditioned if $h\alpha < -2$.

We conclude from this test problem that Euler's method is absolutely stable if h and α are such that

$$-2 < h\alpha < 0. \qquad (3)$$

The set of all values of $h\alpha$ which satisfy this inequality is denoted by $(-2, 0)$ and is called the **interval of absolute stability** for Euler's method.

In *M101*, *Block 1*, the notation $]a, b[$ was used for the open interval $a < x < b$. Here $(-2, 0)$ means the same as $]-2, 0[$

We can now use this analysis to understand what went wrong in Exercise 5. There we tried to solve the differential equation

$$y' = -30y \qquad \text{with } y(0) = 1$$

using Euler's method with a step-size $h = 0.1$. The theory tells us that the method is absolutely stable for this problem only if

$$-2 < -30h < 0,$$

i.e. $h < 1/15$. While the theory does not tell us that we will get accurate answers, for $h = 0.05$ say, it does warn us that it is impossible to even get sensible answers for the step-size we tried, $h = 0.1$.

The tape session in the next section will help you to prove some of the stability results for other methods.

Exercise 6

For what step-sizes is Euler's method absolutely stable for the differential equation

$$y' = -100y \qquad \text{with } y(0) = 1\,?$$

[*Solution on p. 53*]

3.4 Stability in practice

I mentioned in the last subsection that the absolute stability results we obtained for the equation

$$y' = \alpha y \qquad \text{with } y(0) = 1$$

could be extended to more difficult problems Indeed the extension to the general linear differential equation

$$y' = l(x)y + k(x)$$

is quite straightforward. Euler's method gives the recurrence relation

$$\begin{aligned} Y_{r+1} &= Y_r + hY'_r \\ &= Y_r + h[l(x_r)Y_r + k(x_r)] \\ &= [1 + hl(x_r)]Y_r + hk(x_r). \end{aligned}$$

This non-constant-coefficient recurrence relation is absolutely stable (see *Unit 1*, Subsection 3.2) if

$$|1 + hl(x_r)| < 1,$$

i.e. $-2 < hl(x_r) < 0$ for all the values of r.

This is the same as Equation (3) with α replaced by $l(x_r)$.

We would thus have to check, before implementing the method, that this relationship holds for all the values of x_r we are going to use. However, if $l(x_r) > 0$ we can still use Euler's method with some confidence provided that we know that the solution is increasing in magnitude.

The treatment of non-linear equations is not quite so straightforward. We first consider equations of the form

$$y' = m(y).$$

For example

$$y' = 10y\left(1 - \frac{y}{1000}\right) \qquad \text{with } y(0) = 100.$$

(This is the problem considered in the television programme.)

If we apply Euler's method to this type of problem we obtain the recurrence relation

$$Y_{r+1} = Y_r + hm(Y_r). \tag{4}$$

To see whether errors grow, we look at what happens to Y_{r+1} if there is an error, ε, in Y_r, so that the computed value for Y_r is

$$\tilde{Y}_r = Y_r + \varepsilon.$$

Then the computed value for Y_{r+1} is

$$\begin{aligned} \tilde{Y}_{r+1} &= \tilde{Y}_r + hm(\tilde{Y}_r) \\ &= Y_r + \varepsilon + hm(Y_r + \varepsilon). \end{aligned} \tag{5}$$

We can approximate $m(y)$ near Y_r by its Taylor polynomial of degree one as

$$m(Y_r + \varepsilon) \simeq m(Y_r) + \varepsilon \frac{dm}{dy}(Y_r).$$

Hence, using this approximation in Equation (5), we have

$$\tilde{Y}_{r+1} \simeq Y_r + \varepsilon + hm(Y_r) + h\varepsilon \frac{dm}{dy}(Y_r).$$

Using Equation (4) we can reduce this to

$$\tilde{Y}_{r+1} \simeq Y_{r+1} + \varepsilon\left(1 + h\frac{dm}{dy}(Y_r)\right).$$

So the error in the computation of $\tilde{Y}_{r+1}$ is approximately

$$\varepsilon\left(1 + h\frac{dm}{dy}(Y_r)\right)$$

This means that the absolute error, ε, in Y_r has been magnified by a factor $\left(1 + h\frac{dm}{dy}(Y_r)\right)$ in the computation of $\tilde{Y}_{r+1}$. The recurrence relation problem for Euler's method is thus absolutely ill-conditioned if

$$\left|1 + h\frac{dm}{dy}(Y_r)\right| > 1.$$

Hence Euler's method is absolutely stable if

$$-2 < h\frac{dm}{dy}(Y_r) < 0. \tag{6}$$

This is the same as Equation (3) with α replaced by $\frac{dm}{dy}(Y_r)$.

If this condition is true for all values of Y_r in the calculation then Euler's method will be absolutely stable for the differential equation $y' = m(y)$.

Example 2

For the logistic equation considered in the television programme we have

$$m(y) = 10y\left(1 - \frac{y}{1000}\right).$$

Differentiating with respect to y gives

$$\frac{dm}{dy} = 10 - \frac{y}{50}.$$

Now y takes values between 100 and 1000 so that $\frac{dm}{dy}$ takes values between 8 and -10.

Our conclusions would be as follows:

(i) For $100 < Y_r < 500$, $\frac{dm}{dy}(Y_r)$ is positive and so Euler's method is absolutely unstable for any value of h. However, since the solution is increasing rapidly in magnitude, this instability may not cause significant errors in the solution.

(ii) For $500 < Y_r < 1000$, $\frac{dm}{dy}(Y_r)$ is negative and can be as small as -10.

Equation (6) is satisfied with $\frac{dm}{dy} = -10$ only if $h < 0.2$. Thus we would only contemplate using Euler's method with step-size $h < 0.2$.

The above analysis agrees precisely with the results we obtained in the television programme. The instability is most serious for values of Y_r near 1000. In Figure 6 of Section 2 we saw that, for values of h less than 0.2, the sequences generated by the recurrence relation converged to 1000 while, for $h \geq 0.2$, the sequences generated did not converge to 1000. This was caused by the absolute instability of Euler's method for $h \geq 0.2$.

What we have found so far, for non-linear equations of the form

$$y' = m(y), \tag{7}$$

is that the stability condition on the values for h in Euler's method can be obtained by substituting the most negative value of $\frac{dm}{dy}$ for α in the inequality $-2 < h\alpha < 0$.

It can be shown that, for any numerical method applied to the differential equation (7), the stability condition on the values of h can be obtained by substituting the most negative value of $\frac{dm}{dy}$ for α in the inequality which determines the interval of absolute stability for that method.

Finally we consider the general equation

$$y' = m(x, y)$$

with y_0 given.

For example

$$y' = -xy^2$$

with $y(0) = 1$ for $0 \leq x \leq 10$.

If we apply Euler's method to this problem we obtain the recurrence relation

$$Y_{r+1} = Y_r + hm(x_r, Y_r).$$

This can be written as

$$Y_{r+1} = Y_r + hm_r(Y_r) \tag{8}$$

where m_r is the function defined by

$$m_r(y) = m(x_r, y).$$

For $m(x, y) = -xy^2$ at $x_r = 3$ we have $m_r(y) = -3y^2$ as a function of y only.

The treatment of Equation (8) is now identical to that given to Equation (4) and we can deduce that Euler's method is stable if

$$-2 < h\frac{dm_r}{dy}(Y_r) < 0$$

for all values of r.

Hence, when we consider the stability of a numerical method applied to the differential equation

$$y' = m(x, y),$$

we replace α in the inequality which determines the interval of absolute stability by the most negative value of $\frac{dm_r}{dy}(Y_r)$ where $m_r(y) = m(x_r, y)$. (Note that all the previous examples are special cases of this general first-order differential equation.)

Example 3

For the equation

$$y' = -xy^2 \qquad \text{with } y(0) = 1$$

for $0 \leq x \leq 10$ we have

$$m_r(y) = -x_r y^2$$

and $\quad \dfrac{dm_r}{dy}(Y_r) = -2x_rY_r.$

From, for example, a sketch of the direction field for this equation we could show that y lies between 0 and 1. Thus, since $0 \le x \le 10$, the most negative value of $-2x_rY_r$ is -20 and Euler's method is certainly absolutely stable if

$$-2 < -20h < 0$$

i.e. $h < 0.1$.

If we knew the solution we could get a less restrictive bound on h but we have managed to obtain a useful bound on h even without knowing the solution.

From a practical viewpoint, faced with a differential equation, a simple procedure for obtaining numerical solutions to a required accuracy is to

(i) determine bounds, if any, on the values of h to ensure the absolute stability of the method;

(ii) solve the differential equation problem using the numerical method with two different step-sizes, h. A comparison of the results at specific x values will normally indicate the accuracy. If the discrepancies are larger than desired, a smaller step-size should be used and the numerical results compared.

Exercise 7

Euler's method is to be used to solve the differential equation

$$y' = -xy + \frac{1}{1+x^2} \qquad \text{with } y(0) = 1$$

for $0 \le x \le 10$.

Determine a bound on the values of h which will guarantee that Euler's method is absolutely stable.

Exercise 8

Euler's method is to be used to solve the logistic equation

$$y' = 10y\left(1 - \frac{y}{1000}\right)$$

with the initial condition $y(0) = 2000$.

The theory tells us that the solution should tend to the stable population of 1000. Determine a bound on the values of h which will guarantee that Euler's method is absolutely stable

Exercise 9

Euler's method is to be used to solve the differential equation

$$y' = \frac{6x}{y} \qquad \text{with } y(0) = 1.$$

It is known that y takes values between 1 and 5 for $0 \le x \le 2$. Determine a step-size h for which Euler's method will be absolutely stable.

[*Solutions on p. 54*]

Summary of Section 3

1. A one-step numerical method, which uses the recurrence relation

$$Y_{r+1} = Y_r + h\phi(x_r, Y_r, Y_{r+1}, h)$$

is **consistent** with the differential equation

$$y' = m(x, y)$$

if

$$\phi(x_r, y_r, y_r, 0) = m(x_r, y_r).$$

2. The **global error** at x^* is the error

$$Y_{N^*} - y(x^*)$$

where $y(x^*)$ is the true solution at x^* and Y_{N^*} is computed using a recurrence relation N^* times, where

$$x^* = x_0 + N^*h.$$

3. A one-step method is said to be **convergent** for the problem

$$y' = m(x, y) \qquad \text{with } y(x_0) \text{ given},$$

if the global error at x^* tends to zero as the step-size, h, tends to zero, for all points x^* in the interval, $[x_0, b]$, in which approximations are required;

i.e. $\lim_{h \to 0} Y_{N^*} = y(x^*)$,

where $N^*h = x^* - x_0$.

4. A one-step method applied to a well-behaved differential equation is convergent if it is consistent (see Theorem 1). If the local truncation error involves h^{p+1} then the global error at x^*, for small h, is of the form

$$Y_{N^*} - y(x^*) \simeq Ch^p$$

where C does not depend on h.

5. A one-step method, applied to a differential equation, is **absolutely unstable** if the resulting recurrence relation is absolutely ill-conditioned. The method is **absolutely stable** if the recurrence relation problem is absolutely well-conditioned.

6. The **interval of absolute stability** (a, b) is the set of all values of $h\alpha$ for which the method is absolutely stable.

7. For Euler's method we have the following intervals of absolute stability

differential equation	interval of absolute stability
$y' = \alpha y$	$-2 < h\alpha < 0$
$y' = l(x)y + k(x)$	$-2 < hl(x_r) < 0$ for all r
$y' = m(y)$	$-2 < h\dfrac{dm}{dy}(Y_r) < 0$ for all r
$y' = m(x, y)$	$-2 < h\dfrac{dm_r}{dy}(Y_r) < 0$ where $m_r(y) = m(x_r, y)$

For other methods it is sufficient to determine the interval of absolute stability for **the test problem**

$$y' = \alpha y \qquad \text{with } y(0) = 1$$

and then to deduce the appropriate conditions for other differential equations as above.

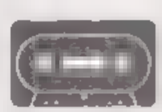
4 Exercises on stability and Simpson's method

4.1 Exercises on stability (Tape Subsection)

In Subsections 3.3 and 3.4 we looked at the absolute stability conditions for Euler's method. In the first part of the tape session we see how the intervals of absolute stability are determined for the trapezoidal method and the predictor-corrector method.

In the second part of the tape we apply these results to the logistic equation of Section 2 to confirm the empirical evidence found in the television programme.

Start the tape when you are ready



1

$$y' = \alpha y \qquad y(0) = 1$$

Recurrence relation for the trapezoidal method

$$Y_{r+1} = Y_r + \frac{h}{2}\left(Y'_r + Y'_{r+1}\right) \tag{1}$$

$$= \boxed{} \tag{1a}$$

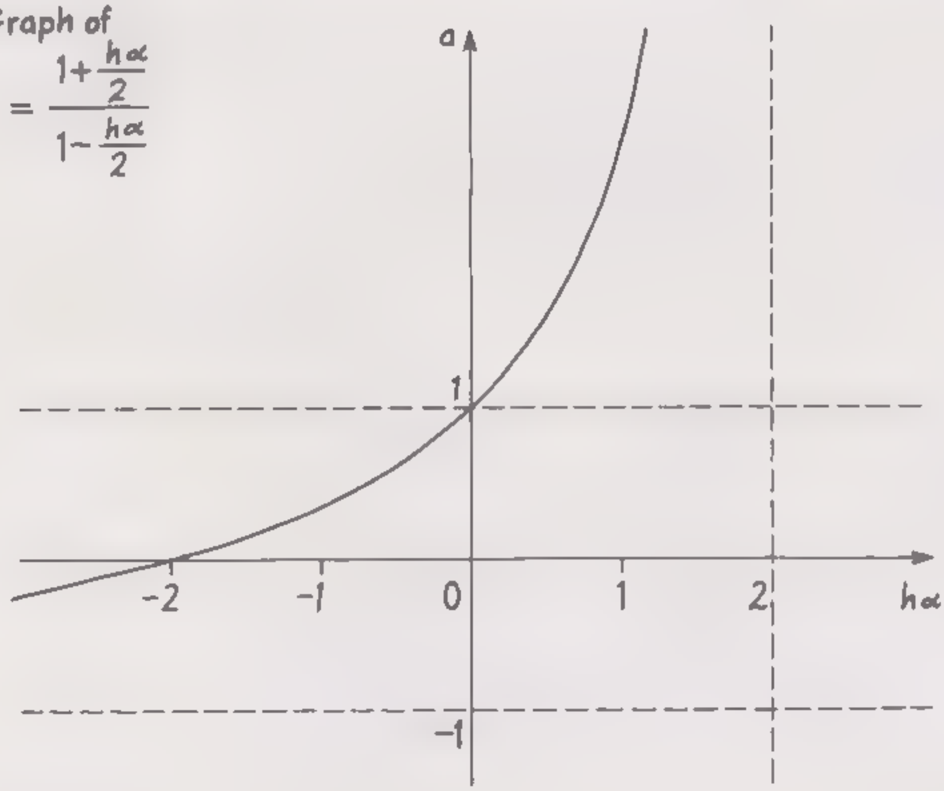
$$\boxed{} = \boxed{} \tag{1b}$$

$$\therefore\ Y_{r+1} = aY_r \qquad \text{where } a = \boxed{}$$

2

Graph of

$$a = \frac{1+\frac{h\alpha}{2}}{1-\frac{h\alpha}{2}}$$



$$|a| < 1 \text{ when } h\alpha < \boxed{} \tag{2a}$$

The interval of absolute stability for the trapezoidal method is $(-\infty, \boxed{})$

3

$$y' = \alpha y \qquad y(0) = 1$$

Predictor-corrector method (Euler-trapezoidal)

a: $Y_r' = \alpha Y_r$

Euler

b: $Y_{r+1}^{*} = Y_r + hY_r'$

$= \boxed{}$

evaluate

c: $Y_{r+1}^{*\prime} = \alpha Y_{r+1}^{*}$

$= \boxed{}$

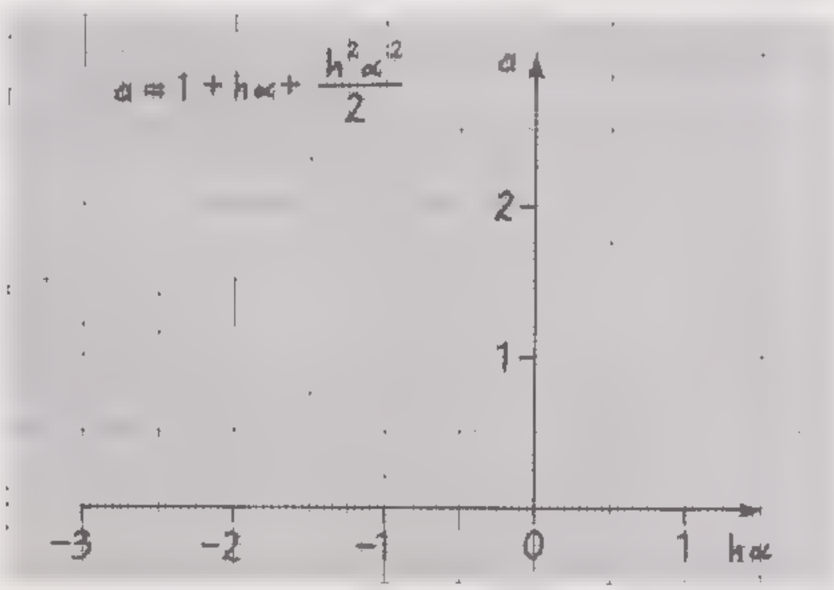
trapezoidal

d: $Y_{r+1} = Y_r + \frac{h}{2}(Y_r' + Y_{r+1}^{*\prime})$

$= \boxed{}$

$$Y_{r+1} = \left(1 + h\alpha + \frac{h^2\alpha^2}{2}\right) Y_r$$

4



$|a| < 1$ when $\boxed{} < h\alpha < \boxed{}$ (4a)

The interval of absolute stability for the predictor-corrector method (Euler-trapezoidal) is $\boxed{}$

5

The logistic equation

$y' = 10y\,(1 - \frac{y}{1000})$ $\qquad y(0) = 100$

$m(y) = \boxed{}$ (5a)

$\frac{dm}{dy} = \boxed{}$ (5b)

If $500 < y < 1000$ the minimum value of $\frac{dm}{dy}$ is $\boxed{}$

6

The trapezoidal method

The interval of absolute stability is $(-\infty, 0)$

$\therefore$ the method is absolutely stable if $h\alpha < 0$.

For the logistic equation, with $500 < y < 1000$, what values of h make the trapezoidal method absolutely stable?

$\boxed{}$

7

The predictor-corrector method (Euler-trapezoidal)

The interval of absolute stability is $(-2, 0)$

The method is absolutely stable if $-2 < h\alpha < 0$

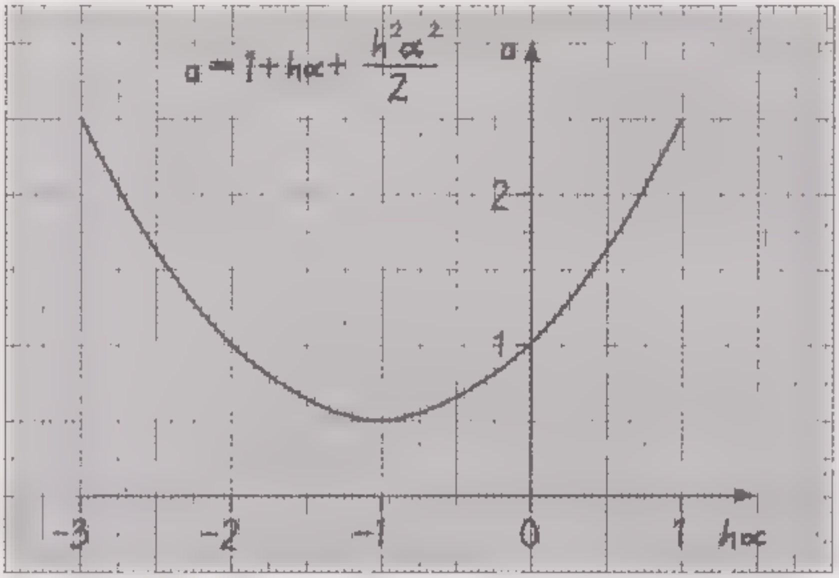
For the logistic equation the method is absolutely stable for $500 < y < 1000$ if

$$-2 < h\frac{dm}{dy} < 0$$

$$-2 < -10h$$

$$\text{i.e. } h < 0{\cdot}2$$

4a



Exercise 1

Find the interval of absolute stability for the Taylor series method of order 2, applied to the test equation $y' = \alpha y$ with $y(0) = 1$, given by

$$Y_{r+1} = Y_r + hY'_r + \frac{h^2}{2}Y''_r$$

where $Y''_r = \alpha Y'_r = \alpha^2 Y_r$.

For what values of h is this method absolutely stable when applied to the logistic equation

$$y' = 10y\left(1 - \frac{y}{1000}\right) \qquad \text{with } y(0) = 500?$$

[*Solution on p. 54*]

4.2 Simpson's method

For simplicity we have only analysed one-step methods so far. Most of the methods used in practice are many-step methods. To give you some idea of how they work we are going to consider a two-step method, Simpson's method. Like the trapezoidal method Simpson's method is an integration method. Because of its high accuracy Simpson's method was very widely used, before the advent of computers, to solve differential equations. It is possible to show that the local truncation error for Simpson's method involves h^5 so that the global error using this method is likely to be much smaller than for most of the methods we've already discussed.

Simpson's method approximates an integral over two steps as

$$\int_{x_{r-1}}^{x_{r+1}} f(x)\,dx \simeq \frac{h}{3}\left(f(x_{r-1}) + 4f(x_r) + f(x_{r+1})\right). \tag{1}$$

We are going to use this formula to solve the differential equation

$$y' = m(x, y). \tag{2}$$

If we integrate both sides of Equation (2) over the interval $[x_{r-1}, x_{r+1}]$ we have

$$\int_{x_{r-1}}^{x_{r+1}} y'\,dy = \int_{x_{r-1}}^{x_{r+1}} m(x, y)\,dx.$$

The left-hand side is just $y_{r+1} - y_{r-1}$ so that, using Simpson's method (1) for the integral on the right-hand side, we have

$$y_{r+1} - y_{r-1} = \int_{x_{r-1}}^{x_{r+1}} m(x,y)\,dx$$

$$\simeq \frac{h}{3}\left(m(x_{r-1},y_{r-1}) + 4m(x_r,y_r) + m(x_{r+1},y_{r+1})\right).$$

This leads us to define the recurrence relation for Simpson's method as

$$Y_{r+1} = Y_{r-1} + \frac{h}{3}(Y'_{r-1} + 4Y'_r + Y'_{r+1})$$

where $Y'_r = m(x_r, Y_r)$.

This is a *second-order* recurrence relation (because it involves terms at x_r and x_{r-1}) and it is also an *implicit* method (because it involves terms in Y'_{r+1} on the right-hand side). This causes two difficulties:

(i) Second-order recurrence relations require two initial values, Y_0 and Y_1, in order to generate the sequence $Y_2, Y_3, Y_4, \ldots$. However, we only have one given initial condition, Y_0, so that somehow we have to generate a value for Y_1 in order to use Simpson's method. This is done by using another method, such as a Taylor series method, to generate Y_1 before switching to Simpson's method.

(ii) Simpson's method cannot be easily used on its own to solve non-linear equations because it is implicit (cf. the trapezoidal method). For this reason Simpson's method is more commonly used as a corrector in a predictor-corrector method.

By a process of algebraic manipulation it is possible to obtain a linear recurrence relation for Simpson's method applied to the linear differential equation

$$y' = l(x)y + k(x)$$

with $y(x_0)$ given.

I will not present those manipulations here but the process is similar to the procedure for the trapezoidal method in Section 1. The result is given in the following procedure box.

Simpson's method for linear differential equations

1. You are given a linear differential equation

$$y' = l(x)y + k(x)$$

and a value for $y(x_0)$.

2. To apply Simpson's method, choose a step-size h and calculate $Y_2, Y_3, \ldots$ from the second-order recurrence relation

$$Y_{r+1} = \left(\frac{4hl_r}{3 - hl_{r+1}}\right)Y_r + \left(\frac{3 + hl_{r-1}}{3 - hl_{r+1}}\right)Y_{r-1} + \left(\frac{h}{3 - hl_{r+1}}\right)(k_{r-1} + 4k_r + k_{r+1}) \qquad (3)$$

where $Y_0 = y(x_0)$, Y_1 has been previously obtained using some other numerical method and $x_r = x_0 + rh$.

$l_r = l(x_r)$
$k_r = k(x_r)$

3. Y_r is an approximation to y_r.

In the following example we use this procedure to illustrate the technique.

Example 1

Use Simpson's method to solve the differential equation

$$y' = x + y \qquad \text{with } y(0) = 0 \qquad (4)$$

using a step-size $h = 0.1$ for $0 \le x \le 1$. Initially generate the value of Y_1 at $x = 0.1$ using a Taylor series method of appropriate order.

Solution

Since Simpson's method has a local truncation error which involves h^5 it is worthwhile computing Y_1 very accurately. The Taylor series method of order n uses the recurrence relation

$$Y_{r+1} = Y_r + hY_r' + \frac{h^2}{2}Y_r'' + \cdots + \frac{h^n}{n!}Y_r^{(n)}$$

where the derivatives may be obtained using the differential equation (4). We have

$$Y_r' = x_r + Y_r, \qquad Y_r'' = 1 + Y_r', \qquad Y_r''' = Y_r'' \qquad \text{and so on.}$$

With $h = 0.1$ we have

$$Y_1 = Y_0 + 0.1Y_0' + \frac{(0.1)^2}{2}Y_0'' + \frac{(0.1)^3}{6}Y_0''' + \cdots + \frac{(0.1)^n}{n!}Y_0^{(n)}$$

where $Y_0 = 0$, $Y_0' = x_0 + Y_0 = 0$, $Y_0'' = 1 + Y_0' = 1$, $Y_0''' = Y_0'' = 1$ and so on.

$$\text{Hence } Y_1 = 0 + 0 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} + \frac{(0.1)^4}{24} + \cdots + \frac{(0.1)^n}{n!}$$

$$= 0.005 + 0.00016667 + 0.00000417 + 0.00000008 + \cdots$$

$$= 0.00517092.$$

The appropriate Taylor series method for eight decimal place accuracy for Y_1 is of order 5.

With this value for Y_1 we use the procedure for Simpson's method. We have

$$l(x) = 1$$

and $\quad k(x) = x.$

Hence the recurrence relation (3) becomes

$$Y_{r+1} = \left(\frac{4h}{3-h}\right)Y_r + \left(\frac{3+h}{3-h}\right)Y_{r-1} + \left(\frac{h}{3-h}\right)(x_{r-1} + 4x_r + x_{r+1}).$$

With $h = 0.1$, and noting that

$$x_{r-1} + x_{r+1} = 2x_r = 2rh,$$

we have

$$Y_{r+1} = \frac{4}{29}Y_r + \frac{31}{29}Y_{r-1} + \frac{1}{29}(0.6r).$$

This second-order recurrence relation can be used to generate the solution using the initial conditions $Y_0 = 0$ and $Y_1 = 0.00517092$. In the following table the true solution

$$y = e^x - x - 1$$

is also given for comparison.

r	x_r	Y_r (Simpson's method)	true solution	error
0	0	0	0	0
1	0.1	0.00517092	0.0051709169	3.1×10^{-9}
2	0.2	0.02140289	0.02140276	1.3×10^{-7}
3	0.3	0.04985897	0.04985881	1.6×10^{-7}
4	0.4	0.09182501	0.09182470	3.1×10^{-7}
5	0.5	0.14872166	0.14872127	3.9×10^{-7}
6	0.6	0.22211938	0.22211880	5.8×10^{-7}
7	0.7	0.31375341	0.31375271	7.0×10^{-7}
8	0.8	0.42554187	0.42554093	9.4×10^{-7}
9	0.9	0.55960425	0.55960311	1.14×10^{-6}
10	1.0	0.71828328	0.71828183	1.42×10^{-6}

You may get slightly different results on your calculator.

All the results generated using Simpson's method are correct to five decimal places. Comparing these results with the results obtained in Example 1 of Section 1 for the same differential equation it can be seen that the results are much better for Simpson's method than for Euler's method, the trapezoidal method or the Taylor series method of order 2. Although quite a lot of work was necessary to compute a value for Y_1, the results clearly make this effort worthwhile if great accuracy is required.

Exercise 2

Use Simpson's method to solve the differential equation

$$y' = 3y + \sin x \qquad \text{with } y(0) = 0.$$

With $h = 0.2$, compute Y_2 and Y_3 given a value $Y_1 = 0.0255$. (Y_1 was obtained using the trapezoidal method with a step-size $h = 0.1$.)

Exercise 3

Show that the use of Simpson's method, with $h = 0.1$, applied to the differential equation

$$y' = 10y \qquad \text{with } y(0) = 1$$

leads to the linear second-order recurrence relation

$$Y_{r+1} = 2Y_r + 2Y_{r-1}.$$

By solving the auxiliary equation

$$x^2 = 2x + 2$$

write down the general solution of the recurrence relation.

[*Solutions on pp. 54–55*]

4.3 The numerical analysis of Simpson's method

As you saw in the last subsection, Simpson's method does give very accurate results provided we obtain a good approximation to Y_1 using some other method. If hundreds or thousands of calculations are to be carried out then Simpson's method is likely to give much better results with fewer calculations than any of the high-order Taylor series methods.

However, there is one further drawback associated with Simpson's method which we hinted at in Exercise 3: Simpson's method cannot be used to solve certain problems. The use of Simpson's method for a linear equation leads to a second-order recurrence relation. From *Unit 1* we know that the general solution of a second-order recurrence relation contains two arbitrary constants. For example the general solution of the recurrence relation in Exercise 3 can be written as

$$Y_n = A \times \lambda^n + B \times \mu^n$$

where $\lambda = 2.7320508$ and $\mu = -0.73205081$.

In *Unit 2* we saw that the general solution of a first-order recurrence relation involves only one arbitrary constant. For example the general solution for the differential equation

$$y' = 10y$$

is $\qquad y = Ae^{10x}.$

This general solution contains only one term while the general solution for Simpson's method contains two terms. What is the correspondence between these two solutions? The general solution to the differential equation can be written as

$$y_r = Ae^{10x_r}$$

and

$$\begin{aligned} y_{r+1} &= Ae^{10x_{r+1}} \\ &= Ae^{10(x_r+h)} \\ &= Ae^{10x_r} \times e^{10h} \\ &= e^{10h} y_r. \end{aligned}$$

Hence there is a recurrence relation for generating y_{r+1} from y_r. With $h = 0.1$ we have

$$y_{r+1} = ey_r$$
$$= 2.7182818y_r.$$

The general solution of this recurrence relation is

$$y_n = A(2.7182818)^n. \qquad (5)$$

Compare this with the solution of the recurrence relation

$$Y_n = A(2.7320508)^n + B(-0.73205081)^n. \qquad (6)$$

There is a similarity between the first term in Y_n and y_n. The second term in Y_n, $B(-0.70205081)^n$, is called a **spurious solution** because it is an extra solution introduced by the method. It does not correspond to anything in the original differential equation. How will this affect the results?

We will apply Simpson's method to the differential equation

$$y' = 10y \qquad \text{with } y(0) = 1.$$

Suppose that we start the Simpson calculations by computing $Y_1 = 2.718$ using some other numerical method. Using this value and $Y_0 = 1$ we can calculate the appropriate values for A and B in (6). The particular solution is

$$Y_n = 0.9959 \times (2.7320508)^n + 0.0041 \times (-0.73205081)^n.$$

The particular solution for the differential equation is obtained by substituting the initial condition, $y_0 = 1$, into Equation (5) giving

$$y_n = (2.7182818)^n.$$

Note that the first term in Y_n is almost identical to the true solution, y_n, while the spurious solution has a small coefficient (0.0041). As n increases the spurious solution gets smaller and smaller since $(-0.73205081)^n$ tends to zero and hence its effect on the solution diminishes. We can conclude that in this case the spurious solution introduced by Simpson's method does not seriously affect the results.

In Exercise 4 you will see that, when Simpson's method with $h = 0.1$ is applied to the differential equation

$$y' = -10y \qquad \text{with } y(0) = 1,$$

the spurious solution *does* seriously affect the results. Without going into the details of the analysis we state that Simpson's method should not be used to solve differential equations of the form

$$y' = l(x)y + k(x)$$

if $l(x)$ is negative for any value of x in the interval under consideration. This is a generalization of the results obtained from Exercises 3 and 4.

Exercise 4

Use Simpson's method to write down the recurrence relation for the differential equation

$$y' = -10y$$

with $y(0) = 1$.

Using a step-size $h = 0.1$ show that the general solution of the recurrence relation is

$$Y_n = A(0.3660254)^n + B(-1.3660254)^n.$$

What is the particular solution with $Y_0 = 1$ and $Y_1 = 0.3679$?

Compare this with the true solution given by

$$y_n = (0.36787944)^n.$$

What happens to Y_n and y_n as n increases?

(You will have an opportunity to see the effect of this spurious solution when you use the computer package.)

[*Solution on p. 55*]

Summary of Section 4

1. The intervals of absolute stability for first-order methods are given in the following table.

method	interval of absolute stability
Euler's method	$(-2, 0)$
trapezoidal method	$(-\infty, 0)$
Taylor series method of order 2	$(-2, 0)$
predictor-corrector method	$(-2, 0)$

2. **Simpson's method** can be used to generate approximate solutions to linear first-order differential equations using the procedure on page 40.

3. Simpson's method can give rise to very accurate solutions since the principal term in its local truncation error involves h^5. However, Simpson's method should not be used to solve linear differential equations of the form

$$y' = l(x)y + k(x)$$

if $l(x)$ is negative as, in this case, the **spurious solution** (the extra solution introduced by the method) will eventually swamp the solution.

5 The computer package NUMSOL

5.1 Introduction

This section is a description of the computer package NUMSOL. The package is designed to solve first-order differential equations using the methods described in this unit. It is intended that you should use this package at Summer School, although it will be available if you want to visit your local computer terminal. A general description of how to input data to packages is given in *Unit 1* Section 6. This package is very similar to the package RECREL described in *Unit 2*.

To help you to decide which method to use for a particular problem I have summarized the properties of each method in the following table:

method	principal term in the l.t.e involves	advantages	disadvantages
Euler's method	h^2	simple to use; explicit method	not very accurate; interval of absolute stability is $(-2, 0)$
Taylor series method of order 2	h^3	explicit method	requires computation of y''. Interval of absolute stability is $(-2, 0)$
predictor-corrector method (Euler-trapezoidal)	h^3	explicit method	interval of absolute stability is $(-2, 0)$
trapezoidal method	h^3	interval of absolute stability is $(-\infty, 0)$	implicit method
Taylor series method of order 3	h^4	explicit method; very accurate	interval of absolute stability is $(-2.51, 0)$* Requires the computation of y'' and y'''.
Simpson's method	h^5	very accurate for some problems	implicit method; should not be used for $y' = l(x)y + k(x)$ if $l(x)$ is negative

* not given in the text earlier.

To illustrate how the package works we begin with an example showing how the logistic equation can be solved at the terminal.

Example 1

Solve the logistic equation

$$y' = 10y\left(1 - \frac{y}{1000}\right) \qquad \text{with } y(0) = 100$$

on the interval $[0, 2]$ using the predictor-corrector method with $h = 0.1$.

Solution

The following terminal dialogue shows how I input this problem, after logging in and obtaining the course library program NUMSOL. Each of my responses is underlined to indicate that this is what I typed. The other characters in the dialogue were output by the computer.

OPTION? 10

Option 10 specifies the differential equation to be solved

Y' = ? 10*Y*(1 − Y/1000)

OPTION? 23

This option specifies the method to be used

PREDICTOR-CORRECTOR METHOD (EULER-TRAPEZOIDAL)

OPTION? 30

INITIAL CONDITION Y(X(0)) = ? 100

OPTION? 32

INITIAL X VALUE X(0) = ? 0

These two options specify the range of x values

OPTION? 33

FINAL X VALUE X(N) = ? 2

OPTION? 34

STEP-SIZE H = ? 0.1

OPTION? SOLVE

Having input all the data, you can solve the problem

5.2 The options for NUMSOL

Before using the package at a terminal you should plan which options you are going to use to solve your problems. A description of each option available for the package is given below.

Command options

OPTIONS — this tells the program to print a list of the available options.

SOLVE — this tells the program to run the problem after all the data has been input. If some data is missing an error message will be printed.

HELP — this tells the program that you are stuck and need advice.

LIST — this tells the program to print the problem and data previously input so that you can check the problem you are solving.

STOP — this is the only way to stop the program.

ANSWER — this gives the solution to the computing exercises so that you can check that your answer is correct. To the question

EXERCISE?

you respond with the number of the exercise you are interested in.

Problem options

10 **Input $m(x, y)$**

To the question

Y' = ?

you respond with a valid input expression for $m(x, y)$.

Valid input expressions are described in *Unit 1*.

For linear equations you must input

$l(x) \times y + k(x)$

in that order so that the package can recognize the linearity.

11 **—Order and derivatives for Taylor series method**

To the question

ORDER = ?

you respond with the order (≤ 5) of the Taylor series method. When you have specified the order the program will ask you for each of the higher derivatives that the method requires.

For example if you want to use a Taylor series method of order 2 for the differential equation

$y' = xy + 3$

you would respond to ORDER = ? with 2. The package would then ask you for the second derivative

Y'' = ?

to which you would respond Y + X * Y'

(The differential equation would have been input using Option 10 so Y' is already known.)

12 **—Change order of Taylor series method**

If you wish to change the order of the Taylor series method without having to input previous derivatives again, you would use this option. To the question

ORDER = ?

you would respond as in Option 11 and if further derivatives are required you will be asked for them.

For example if you use the same differential equation as in the example in Option 11 but wish to change to the Taylor series method of order 3, to the question ORDER = ? you would respond with 3. The package already has Y' (Option 10) and Y'' (Option 11) and so only Y''' is missing. The package will ask for this and to the question

Y''' = ?

you would respond 2 * Y' + X * Y''

Note that only single quotes, repeated as necessary, are used for derivatives e.g. Y'' *not* Y"

Method options

20 **—Euler's method**

This uses a first-order recurrence relation to solve the differential equation. The initial y value is input using Option 30

21 **—Taylor series method**

The order of the Taylor series method and any further derivatives required are specified using Option 11 and the order can be changed using Option 12. The method uses a first-order recurrence relation to solve the differential equation.

22 **—Trapezoidal method**

This method can only be used to solve linear differential equations using a first-order recurrence relation.

23 —**Predictor-corrector method**

This method uses Euler's method to predict and the trapezoidal method to correct. Only one initial y value needs to be specified (Option 30).

24 —**Simpson's method**

This method can only be used to solve linear differential equations. The method involves the use of a second-order recurrence relation for which two initial y values are required (Options 30 and 31). Y_1 may be computed using a one-step method with small step-size if it is not known.

Data options

30 —**Initial condition Y_0**

This specifies the initial y value.

31 —**Initial condition Y_1**

This specifies the second initial condition for Simpson's method.

32 —**x_0**

—This is used to input the initial x value.

33 —**x_N**

This is used to input the final x value.

34 —**The step-size h**

This parameter specifies the step-size for the numerical method.

Print options

40 —**Outline printout**

If you only want selected output you should use this option to specify k such that every kth term in the sequence is printed.

41 —**Full printout (default)**

All terms in the sequence will be printed.

A default option is the one the program will use if no other is specified.

42 —**Print solution only**

Only the last term in the sequence is printed. (Intermediate terms will not be printed.)

5.3 Computer exercises

The following exercises will require the use of the computer package Before you run the exercises at a terminal you should plan how you are going to tackle the problems you wish to solve Judicious preliminary work will enable you to make the best use of your limited time at the terminal. You may not have time to solve all these problems.

Exercise 1: This exercise will reproduce the results of the television programme.

Use the package to solve the logistic equation

$$y' = 10y\left(1 - \frac{y}{1000}\right) \qquad \text{with } y(0) = 100$$

for $0 \leq x \leq 1$,

by each of the following methods

(i) Euler's method,
(ii) predictor-corrector method,
(iii) Taylor series method of order 2.

For each method use a step-size $h = 0.1$.

If you have time, look at what happens for larger step-sizes.

Exercise 2: To investigate stability

The differential equation

$$y' = -xy + \frac{1}{1+x^2} \qquad \text{with } y(0) = 1$$

is to be solved on the interval $[0, 10]$ using each of the following methods

(i) Euler's method
(ii) trapezoidal method
(iii) Simpson's method

with step-sizes $h = 1$, 0.5 and 0.1.

(To reduce output, print the results only at the integer values of x.)

Do the results agree with the stability analysis in Sections 3 and 4?

Exercise 3: To investigate accuracy

What methods can be used to solve the differential equation

$$y' = -e^y \qquad \text{with } y(0) = 1$$

on the interval $[0, 1]$?

The valid expression for e^y is EXP(Y)

Compare the numerical results obtained using Taylor series methods of order 2 and order 3 and a step-size of $h = 0.1$. To get more accurate results use $h = 0.05$.

$\left(\text{Hint: } \dfrac{d}{dx}e^y = y'e^y = -e^{2y}.\right)$

Is it more accurate, for this problem, to use a step-size $h = 0.05$ with a Taylor series method of order 2 or to use a step-size $h = 0.1$ with a Taylor series method or order 3?

Exercise 4: To investigate stability

Experiment with the differential equation

$$y' = -30y \qquad \text{with } y(0) = 1$$

for $0 \le x \le 1$, to test the stability of the methods in the package for different values of h. The stability of Euler's method for this equation is discussed in Section 3.

Exercise 5: To investigate stability

Use Simpson's method with $h = 0.1$ to solve the differential equation

$$y' = -10y \qquad \text{with } y(0) = 1$$

for $0 \le x \le 2$.

Confirm the results of Exercise 4 in Section 4.

(Take $Y_1 = 0.3679$ and then $Y_1 = 0.3660$.)

[*Solutions to the computer exercises are not given in the unit. A check solution can be obtained at the terminal using the option* ANSWER]

6 End of unit exercises

Exercise 1

Compute the first 3 terms in the sequence if the trapezoidal method is to be used with a step-size $h = 0.1$ to solve the differential equation

$$y' = y + 2x \qquad \text{with } y(0) = 1.$$

Exercise 2

The predictor-corrector method is to be used with $h = 0.1$ to solve the equation

$$y' = 7y \qquad \text{with } y(0) = 1.$$

Write down a single recurrence relation which can be used to generate the sequence. What is the particular solution for this recurrence relation?

Exercise 3

Is the recurrence relation $Y_{r+1} = Y_r + \frac{h}{5}(4Y'_r + Y'_{r+1})$ consistent with the differential equation

$$y' = m(x, y)\,?$$

Would you expect this method to give better results than the trapezoidal method for small values of h?

Exercise 4

For what values of $h\alpha$ is the method in Exercise 3 absolutely stable when applied to the problem

$$y' = \alpha y$$

with $y(0) = 1$?

Exercise 5

Simpson's method is to be used to solve the linear differential equation

$$y' = 3y + 2x$$

with $y(0) = 1$.

Derive the recurrence relation which can be used to generate the solution for a given step-size h.

What other information do you require and how would you get it?

[*Solutions on pp. 55–56*]

Appendix

Solutions to the exercises in Section 1

1. The trapezoidal method utilizes the recurrence relation

$$Y_{r+1} = Y_r + \frac{h}{2}(Y'_r + Y'_{r+1})$$

For the differential equation

$$y' = 3y + \sin x$$

we have

$$Y_{r+1} = Y_r + \frac{h}{2}[(3Y_r + \sin x_r) + (3Y_{r+1} + \sin x_{r+1})].$$

Collecting up the terms gives

$$Y_{r+1}\left(1 - \frac{3h}{2}\right) = Y_r\left(1 + \frac{3h}{2}\right) + \frac{h}{2}(\sin x_r + \sin x_{r+1})$$

i.e.

$$Y_{r+1} = \frac{(1 + 3h/2)Y_r + (h/2)(\sin x_r + \sin x_{r+1})}{1 - 3h/2}$$

[Alternatively we could just use the procedure box to write down the recurrence relation directly.]

With $h = 0.2$ we have

$$Y_{r+1} = \frac{1.3Y_r + 0.1(\sin x_r + \sin x_{r+1})}{0.7}.$$

The figures in the following table are quoted to 5 decimal places.

r	0	1	2	3
x_r	0	0.2	0.4	0.6
Y_r	0	0.02838	0.13672	0.39020
y_r	0	0.02460	0.12308	0.35304
$Y_r - y_r$	0	0.00378	0.01364	0.03716

Whilst the results are not particularly accurate, because of the rather large step-size, the 'shape' of the approximate solution is correct. The maximum error is approximately 0.04 at $x = 0.6$.

2. The formula for the trapezoidal method is

$$Y_{r+1} = Y_r + \frac{h}{2}(Y'_r + Y'_{r+1}).$$

For the non-linear differential equation

$$y' = \sin y$$

we have

$$Y_{r+1} = Y_r + \frac{h}{2}(\sin Y_r + \sin Y_{r+1}).$$

Collecting the terms in Y_{r+1} and Y_r we have

$$Y_{r+1} - \frac{h}{2}\sin Y_{r+1} = Y_r + \frac{h}{2}\sin Y_r.$$

Putting $h = 0.1$ gives

$$Y_{r+1} - 0.05\sin Y_{r+1} = Y_r + 0.05\sin Y_r.$$

With $y(0) = Y_0 = 1$, to get Y_1 we need to solve the equation

$$Y_1 - 0.05\sin Y_1 = 1.04207355$$

The solution cannot be written down immediately and the Newton-Raphson method, as described in *Unit 18*, Section 2, would have to be used to determine Y_1. This method involves a considerable amount of extra work at each step.

3. Assuming that Y_r lies on a solution curve so that

$$Y_r = y_r, \quad Y_r' = y_r' \quad \text{and} \quad Y_r'' = y_r''$$

we have, for the Taylor series method of order 2,

$$Y_{r+1} = y_r + hy_r' + \frac{h^2}{2}y_r''.$$

The true solution y_{r+1} can be determined using a Taylor series expansion at x_r as

$$y_{r+1} = y_r + hy_r' + \frac{h^2}{2}y_r'' + \frac{h^3}{6}y_r''' + \frac{h^4}{24}y_r^{(iv)} + \cdots$$

Hence the local truncation error is

$$Y_{r+1} - y_{r+1} = -\frac{h^3}{6}y_r''' - \frac{h^4}{24}y_r^{(iv)} - \cdots$$

The principal term in the local truncation error is

$$-\frac{h^3}{6}y_r'''$$

4. Assuming that Y_r lies on a solution curve so that

$$Y_r = y_r, \quad Y_r' = y_r', \quad Y_r'' = y_r'' \quad \text{and} \quad Y_r''' = y_r'''$$

we have, for the Taylor series method of order 3,

$$Y_{r+1} = y_r + hy_r' + \frac{h^2}{2}y_r'' + \frac{h^3}{6}y_r'''.$$

The true solution satisfies

$$y_{r+1} = y_r + hy_r' + \frac{h^2}{2}y_r'' + \frac{h^3}{6}y_r''' + \frac{h^4}{24}y_r^{(iv)} + \frac{h^5}{120}y_r^{(v)} + \cdots$$

Hence the local truncation error is

$$Y_{r+1} - y_{r+1} = -\frac{h^4}{24}y_r^{(iv)} - \frac{h^5}{120}y_r^{(v)} - \cdots.$$

The principal term in the local truncation error is

$$-\frac{h^4}{24}y_r^{(iv)}.$$

If the step-size is halved so that $h^* = h/2$ the principal term in the local truncation error becomes

$$-\frac{(h^*)^4}{24}y_r^{(iv)} = -\frac{(h/2)^4}{24}y_r^{(iv)} = -\frac{h^4}{384}y_r^{(iv)}.$$

The local truncation error has been reduced by a factor of approximately 16. Note however that about twice as many calculations would be needed to obtain an approximation at a particular value of x so that we would not expect the accumulated error at this value of x to be reduced by a factor of 16

5. The Taylor series method of order n is given by

$$Y_{r+1} = Y_r + hY_r' + \frac{h^2}{2}Y_r'' + \cdots + \frac{h^n}{n!}Y_r^{(n)}$$

Assuming that Y_r lies on a solution curve so that

$$Y_r = y_r, \quad Y_r' = y_r', \quad \ldots, \quad Y_r^{(n)} = y_r^{(n)}$$

we have

$$Y_{r+1} = y_r + hy_r' + \frac{h^2}{2}y_r'' + \cdots + \frac{h^n}{n!}y_r^{(n)}.$$

The true solution satisfies

$$y_{r+1} = y_r + hy_r' + \frac{h^2}{2}y_r'' + \cdots + \frac{h^n}{n!}y_r^{(n)} + \frac{h^{n+1}}{(n+1)!}y_r^{(n+1)} + \frac{h^{n+2}}{(n+2)!}y_r^{(n+2)} + \cdots$$

Hence the local truncation error is

$$Y_{r+1} - y_{r+1} = -\frac{h^{n+1}}{(n+1)!}y_r^{(n+1)} - \frac{h^{n+2}}{(n+2)!}y_r^{(n+2)} - \cdots$$

6. Here is the table of errors, for $h = 0.1$, using the Taylor series method of order 2 to solve the differential equation

$$y' = x + y$$

with $y(0) = 1$.

x_r	Y_r	y_r	$Y_r - y_r$
0	0	0	0
0.1	0.005	0.005171	−0.000171
0.2	0.021025	0.021403	−0.000378
0.3	0.049232	0.049859	−0.000627

For $h = 0.05$ we have the following table:

x_r	Y_r	y_r	$Y_r - y_r$
0	0	0	0
0.1	0.005127	0.005171	−0.000044
0.2	0.021305	0.021403	−0.000098
0.3	0.049696	0.049859	−0.000163

At $x = 0.1$ the errors are -0.000171 and -0.000044. Hence the error has been reduced by a factor of $171/44 \simeq 3.89$. Similarly at $x = 0.2$ and 0.3 the errors have been reduced by factors of approximately 3.86 and 3.85 respectively.

We can conclude that, for the Taylor series method of order 2, halving the step-size reduces the errors at particular values of x by a factor of approximately 4. (Note that if we halve the step-size the local truncation error is reduced by a factor of approximately 8.)

7. Assuming that the values on the right-hand side are correct, we have

$$Y_{r+1} = y_r + hy_{r+1}'.$$

The Taylor series expansion for y_{r+1}' is given by

$$y_{r+1}' = y_r' + hy_r'' + \frac{h^2}{2}y_r''' + \frac{h^3}{6}y_r^{(iv)} + \cdots.$$

$$\therefore Y_{r+1} = y_r + hy_r' + h^2y_r'' + \frac{h^3}{2}y_r''' + \cdots.$$

The Taylor series expansion for y_{r+1} is

$$y_{r+1} = y_r + hy_r' + \frac{h^2}{2}y_r'' + \frac{h^3}{6}y_r''' + \cdots.$$

Hence

$$Y_{r+1} - y_{r+1} = \frac{h^2}{2}y_r'' + \frac{h^3}{3}y_r''' + \cdots .$$

The principal term in the local truncation error is

$$\frac{h^2}{2}y_r''.$$

This term involves h^2, as does Euler's method

8. (i) The principal term in the local truncation error for the trapezoidal method is

$$\frac{h^3}{12}y_r'''.$$

If we halve the step-size, putting $h^* = h/2$, the principal term in the local truncation error becomes

$$\frac{h^{*3}}{12}y_r''' = \frac{1}{8} \times \frac{h^3}{12}y_r'''.$$

Hence the local truncation error will reduce by a factor of approximately 8.

The Taylor series method of order 2 also has a local truncation error whose principal term involves h^3 and so halving the step-size would reduce the local truncation error by a factor of approximately 8 for this method too.

(ii) From the results of Exercise 6, for the Taylor series method of order 2, we would expect the accumulated errors for the trapezoidal method to reduce by a factor of approximately 4 if the step-size were halved.

9. At $x = 0.1$ the error has been reduced by a factor of $92/23 = 4.00$. Similarly at $x = 0.2$ and $x = 0.3$ the errors have been reduced by factors of 4.00 and 4.02 respectively.

We conclude that, for this problem, halving the step-size reduces the errors at particular values of x by a factor of approximately 4 for the trapezoidal method.

Solutions to the exercises in Section 2

1. Separating the variables (see *Unit 2*) we have

$$\frac{y'}{y(1-y/1000)} = 10. \qquad (1)$$

To get the left-hand side into the required form we use partial fractions to write

$$\frac{1}{y(1-y/1000)} = \frac{A}{y} + \frac{B}{(1-y/1000)}.$$

This leads to the two equations for A and B as

$$A = 1,$$

$$\frac{-A}{1000} + B = 0.$$

Hence Equation (1) becomes

$$\frac{y'}{y(1-y/1000)} = \frac{y'}{y} + \frac{y'}{1000-y} = 10.$$

Integrating both sides of Equation (1) gives

$$\log y - \log(1000 - y) = 10x + C,$$

i.e.

$$\log\left(\frac{y}{1000-y}\right) = 10x + C$$

giving

$$\frac{y}{1000-y} = Ae^{10x}.$$

This simplifies to

$$y = \frac{1000\,Ae^{10x}}{1 + Ae^{10x}}.$$

For the initial condition $y(0) = 100$ we have

$$100 = \frac{1000\,A}{1 + A}.$$

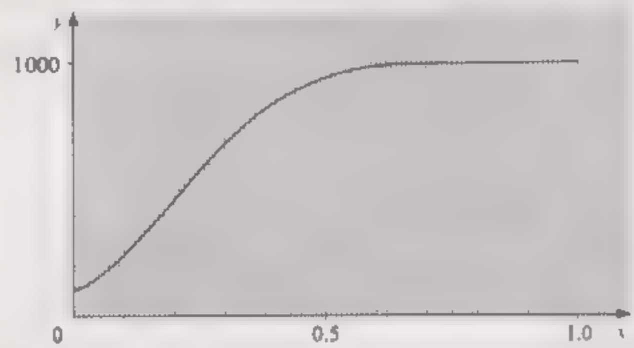
Hence $A = \frac{1}{9}$.

The particular solution is

$$y = \frac{1000e^{10x}}{9 + e^{10x}},$$

i.e. $y = \dfrac{1000}{9e^{-10x} + 1}$.

Here is a sketch of this solution.



2. Using the predictor-corrector method to evaluate Y_3 and Y_4 we proceed through the four steps of the algorithm:

(i) *Evaluate* $Y_2' = 10Y_2\left(1 - \dfrac{Y_2}{1000}\right) = 2447.79$
(using the differential equation).

(ii) *Predict* $Y_3^* = Y_2 + hY_2' = 672.52255$
(Euler's method).

(iii) *Evaluate* $Y_3^{*\prime} = 10Y_3^*\left(1 - \dfrac{Y_3^*}{1000}\right) = 2202.3597$
(using the differential equation).

(iv) *Correct* $Y_3 = Y_2 + \dfrac{h}{2}(Y_2' + Y_3^{*\prime}) = 660.25103$
(trapezoidal method).

Hence our approximation at $x = 0.3$ is $Y_3 = 660.25103$. Repeating the steps for Y_4 we have:

(i) *Evaluate* $Y_3' = 10Y_3\left(1 - \dfrac{Y_3}{1000}\right) = 2243.1961$

(ii) *Predict* $Y_4^* = Y_3 + hY_3' = 884.57064.$

(iii) *Evaluate* $Y_4^{*\prime} = 10Y_4^*\left(1 - \dfrac{Y_4^*}{1000}\right) = 1021.0542.$

(iv) *Correct* $Y_4 = Y_3 + \dfrac{h}{2}(Y_3' + Y_4^{*\prime}) = 823.46355.$

Hence our approximation at $x = 0.4$ is $Y_4 = 823.46355$.

3. For the differential equation

$$y' = \log y + x \qquad \text{with } y(0) = 1$$

and $h = 0.1$, the predictor-corrector method gives:

(i) *Evaluate* $Y_0' = \log Y_0 + x_0 = \log 1 + 0 = 0.$
(ii) *Predict* $Y_1^* = Y_0 + hY_0' = 1.$
(iii) *Evaluate* $Y_1^{*\prime} = \log Y_1^* + x_1 = 0.1.$
(iv) *Correct* $Y_1 = Y_0 + \dfrac{h}{2}(Y_0' + Y_1^{*\prime}) = 1.005.$

Hence $Y_1 = 1.005$ at $x_1 = 0.1$.

(i) *Evaluate* $Y_1' = \log Y_1 + x_1 = 0.10498754.$
(ii) *Predict* $Y_2^* = Y_1 + hY_1' = 1.0154988.$
(iii) *Evaluate* $Y_2^{*\prime} = \log Y_2^* + x_2 = 0.21537988.$
(iv) *Correct* $Y_2 = Y_1 + \dfrac{h}{2}(Y_1' + Y_2^{*\prime}) = 1.0210184.$

Hence $Y_2 = 1.021$ at $x = 0.2$.

4. For the differential equation

$$y' = \sin y$$

with $y(0) = 1$ and $h = 0.2$ the predictor-corrector method gives the following approximations at $x = 0.2$ and $x = 0.4$:

(i) *Evaluate* $Y_0' = \sin Y_0 = 0.84147099.$
(ii) *Predict* $Y_1^* = Y_0 + hY_0' = 1.1682942.$
(iii) *Evaluate* $Y_1^{*\prime} = \sin Y_1^* = 0.92008374.$
(iv) *Correct* $Y_1 = Y_0 + \dfrac{h}{2}(Y_0' + Y_1^{*\prime}) = 1.1761555.$

Hence $Y_1 = 1.176$ at $x = 0.2$.

(i) *Evaluate* $Y_1' = \sin Y_1 = 0.92313471.$
(ii) *Predict* $Y_2^* = Y_1 + hY_1' = 1.3607824.$
(iii) *Evaluate* $Y_2^{*\prime} = \sin Y_2^* = 0.97802802.$
(iv) *Correct* $Y_2 = Y_1 + \dfrac{h}{2}(Y_1' + Y_2^{*\prime}) = 1.3662717.$

Hence $Y_2 = 1.366$ at $x = 0.4$.

5. $Y_{r+1}^* = Y_r + hY_r' = Y_r + hm(x_r, Y_r)$ using the differential equation

$$Y_{r+1} = Y_r + \frac{h}{2}(Y_r' + Y_{r+1}^{*\prime})$$

$$= Y_r + \frac{h}{2}[m(x_r, Y_r) + m(x_{r+1}, Y_{r+1}^*)].$$

Substituting for Y_{r+1}^* in this second equation gives

$$Y_{r+1} = Y_r + \frac{h}{2}[m(x_r, Y_r) + m(x_r, Y_r + hm(x_r, Y_r))]$$

As this does not have Y_{r+1}' on the right-hand side the predictor-corrector method is explicit.

Solutions to the exercises in Section 3

1. (i) $\phi(x_r, Y_r, Y_{r+1}, h) = \dfrac{1}{3}(Y_r' + 2Y_{r+1}').$

Hence

$$\phi(x_r, y_r, y_r, 0) = \frac{1}{3}(y_r' + 2y_r')$$

$$= y_r' = m(x_r, y_r).$$

Thus the method is consistent with the differential equation.

(ii) $\phi(x_r, Y_r, Y_{r+1}, h) = \dfrac{1}{4}(3Y_r' - Y_{r+1}').$

Hence

$$\phi(x_r, y_r, y_r, 0) = \frac{1}{4}(3y_r' - y_r')$$

$$= \frac{1}{2}y_r' = \frac{1}{2}m(x_r, y_r).$$

Thus the method is not consistent with the differential equation.

2. $\phi(x_r, Y_r, Y_{r+1}, h) = aY_r' + bY_{r+1}'.$

Hence

$$\phi(x_r, y_r, y_r, 0) = ay_r' + by_r'$$

$$= (a + b)m(x_r, y_r).$$

The method is consistent with the differential equation

$$y' = m(x, y)$$

only if $a + b = 1$.

If $a = b = \frac{1}{2}$ we have the trapezoidal method. If $a = 1$, $b = 0$ we have Euler's method.

3. From Exercise 2 with $a = b = \frac{1}{2}$ we know that the trapezoidal method is consistent. From Theorem 1 the method is also convergent. Since the principal term in the local truncation error for the trapezoidal method involves h^3, Theorem 1 tells us that the global error at x^* is of the form

$$Y_{N^*} - y(x^*) \simeq Ch^2$$

for small values of h. Thus halving the step-size would reduce the global error at x^* by a factor of approximately 4.

4. From Exercise 5 in Section 2 we have

$$Y_{r+1} = Y_r + \frac{h}{2}[m(x_r, Y_r) + m(x_r, Y_r + hm(x_r, Y_r))].$$

Hence

$$\phi(x_r, Y_r, Y_{r+1}, h) = \tfrac{1}{2}[m(x_r, Y_r) + m(x_r, Y_r + hm(x_r, Y_r))].$$

Thus

$$\phi(x_r, y_r, y_r, 0) = \tfrac{1}{2}[m(x_r, y_r) + m(x_r, y_r)]$$

$$= m(x_r, y_r).$$

The predictor-corrector method is consistent with the differential equation.

5. Euler's method uses the recurrence relation

$$Y_{r+1} = Y_r + hY_r'.$$

For the differential equation

$$y' = -30y$$

with $y(0) = 1$ and $h = 0.1$ we have

$$Y_{r+1} = Y_r - 30 \times 0.1 Y_r$$

$$= -2Y_r.$$

With $Y_0 = 1$ we have the particular solution for this recurrence relation as

$$Y_n = (-2)^n.$$

This is an increasing oscillating solution. (The sequence generated is 1, −2, 4, −8, 16, −32, 64...)

The true solution is $y = e^{-30x}$ which tends rapidly to zero. Something has clearly gone wrong with the method.

6. Euler's method is absolutely stable if $-2 < h\alpha < 0$. Hence for the differential equation

$$y' = -100y$$

$\alpha = -100$ giving

$$-2 < -100h < 0$$

i.e. $100h < 2$. (The inequality $-100h < 0$ is always satisfied for positive h.)

i.e. $h < 0.02$.

7. The differential equation

$$y' = -xy + \frac{1}{1+x^2}$$

is linear with $l(x) = -x$ and $k(x) = \dfrac{1}{1+x^2}$.

The condition for absolute stability in Euler's method is that

$$-2 < l(x)h < 0$$

The minimum value for $l(x)$ is -10 since $0 \le x \le 10$.

Hence we require

$$-2 < -10h$$

i.e. $10h < 2$

$$h < 0.2.$$

8. The theory tells us that y lies between 1000 and 2000 in the solution of the logistic equation

$$y' = 10y\left(1 - \frac{y}{1000}\right) \qquad \text{with } y(0) = 2000.$$

Hence the minimum value of

$$\frac{dm}{dy} = 10 - \frac{y}{50}$$

occurs when $y = 2000$ giving $\dfrac{dm}{dy} = -30$

We require

$$-2 < h\frac{dm}{dy} < 0$$

i.e. $30h < 2$

$$h < \frac{1}{15}.$$

Thus, provided $h < 1/15$, Euler's method is absolutely stable.

9. This is a differential equation of the form

$$y' = m(x, y).$$

At x_r we write

$$m_r(y) = m(x_r, y_r).$$

In this case $m(x, y) = \dfrac{6x}{y}$

so that

$$m_r(y) = \frac{6x_r}{y}.$$

Now

$$\frac{dm_r}{dy} = -\frac{6x_r}{y^2}$$

The condition for absolute stability in Euler's method is

$$-2 < h\frac{dm_r}{dy} < 0.$$

x lies between 0 and 2 whilst y lies between 1 and 5. Hence the minimum value of $\dfrac{dm_r}{dy}$ occurs when $x_r = 2$ and $y = 1$ giving

$$\frac{dm_r}{dy} \geq -12$$

Hence the method is certainly absolutely stable if

$$-2 < -12h.$$

i.e. $h < \dfrac{1}{6}$.

(In reality we could get a much less restrictive bound on h by obtaining the solution using the separation of variables method and then finding the minimum value of $-6x/y^2$, but that would defeat the object of the exercise!)

Solutions to the exercises in Section 4

1. For the Taylor series method of order 2 we have

$$Y_{r+1} = Y_r + hY'_r + \frac{h^2}{2}Y''_r. \qquad (1)$$

Applied to the problem

$$y' = \alpha y$$

we have $Y'_r = \alpha Y_r$ and $Y''_r = \alpha Y'_r = \alpha^2 Y_r$ (differentiating the differential equation). Substituting into Equation (1) above gives

$$Y_{r+1} = Y_r + h\alpha Y_r + \frac{h^2\alpha^2}{2}Y_r$$

$$= \left(1 + h\alpha + \frac{h^2\alpha^2}{2}\right)Y_r$$

This is a recurrence relation of the form

$$Y_{r+1} = aY_r$$

where

$$a = 1 + h\alpha + \frac{h^2\alpha^2}{2}.$$

Note that this is the same equation for a that we had in the predictor-corrector method in tape frame 3. Hence the graph will be the same as in tape frame 4 and consequently the interval of absolute stability ($|a| < 1$) is $(-2, 0)$.

In the logistic equation, with $y(0) = 500$ the solution increases rapidly to 1000 and $\dfrac{dm}{dy}$ lies between 0 and -10 where

$$m(y) = 10y\left(1 - \frac{y}{1000}\right).$$

As the interval of absolute stability for this method is $(-2, 0)$ the method is stable provided $h\dfrac{dm}{dy} > -2$, i.e. $h < 0.2$.

2. Using the procedure for Simpson's method applied to linear differential equations with $l(x) = 3$, $k(x) = \sin x$ and $h = 0.2$ we obtain the recurrence relation

$$Y_{r+1} = \frac{2.4}{2.4}Y_r + \frac{3.6}{2.4}Y_{r-1} + \frac{0.2}{2.4}(\sin x_{r-1} + 4\sin x_r + \sin x_{r+1})$$

$$= Y_r + 1.5Y_{r-1} + \frac{1}{12}(\sin x_{r-1} + 4\sin x_r + \sin x_{r+1})$$

Using the initial values $Y_0 = 0$ at $x_0 = 0$ and $Y_1 = 0.0255$ at $x_1 = 0.2$ we have:

$$Y_2 = 0.0255 + 1.5 \times 0 + \frac{1}{12}(\sin 0 + 4\sin 0.2 + \sin 0.4)$$

$$= 0.12417464,$$

$$Y_3 = 0.12417464 + 1.5 \times 0.0255$$

$$+ \frac{1}{12}(\sin 0.2 + 4\sin 0.4 + \sin 0.6)$$

$$= 0.35584007.$$

3. Using the procedure for linear equations with $l(x) = 10$, $k(x) = 0$ and $h = 0.1$ we obtain the recurrence relation as

$$Y_{r+1} = \left(\frac{4 \times 0.1 \times 10}{3 - 0.1 \times 10}\right) Y_r + \left(\frac{3 + 0.1 \times 10}{3 - 0.1 \times 10}\right) Y_{r-1} + 0$$
$$= 2Y_r + 2Y_{r-1}.$$

This is the required recurrence relation.

The auxiliary equation (cf. *Unit 1*, Section 2) is

$$x^2 = 2x + 2 \qquad \text{or} \qquad x^2 - 2x - 2 = 0.$$

The roots of this quadratic are

$$\lambda, \mu = \frac{2 \pm \sqrt{4 + 8}}{2}$$
$$= 1 \pm \sqrt{3}.$$

Thus $\lambda = 1 + \sqrt{3} = 2.7320508$ and
$\mu = 1 - \sqrt{3} = -0.73205081$.

The general solution for this recurrence relation is

$$Y_n = A \times \lambda^n + B \times \mu^n$$

where $\lambda = 2.7320508$ and $\mu = -0.73205081$.

4. Again using the procedure for linear equations with $l(x) = -10$, $k(x) = 0$ and $h = 0.1$ we have

$$Y_{r+1} = \left(\frac{4 \times 0.1 \times (-10)}{3 - 0.1 \times (-10)}\right) Y_r + \left(\frac{3 + 0.1 \times (-10)}{3 - 0.1 \times (-10)}\right) Y_{r-1} + 0$$
$$= -Y_r + \tfrac{1}{2} Y_{r-1}.$$

This is a linear, constant-coefficient, homogeneous, second-order recurrence relation. Its auxiliary equation is

$$x^2 = -x + \tfrac{1}{2} \qquad \text{or} \qquad x^2 + x - \tfrac{1}{2} = 0.$$

The roots of this equation are

$$\lambda, \mu = \frac{-1 \pm \sqrt{1 + 2}}{2};$$

i.e. $\lambda = \dfrac{-1 + \sqrt{3}}{2} = 0.3660254$ and

$$\mu = \frac{-1 - \sqrt{3}}{2} = -1.3660254.$$

Hence the general solution is

$$Y_n = A(0.3660254)^n + B(-1.3660254)^n.$$

If $Y_0 = 1$, and $Y_1 = 0.3679$ the particular solution is obtained from solving for A and B in

$$Y_0 = A + B = 1$$
$$Y_1 = 0.3660254A - 1.3660254B = 0.3679.$$

This gives $A = 1.0010823$ and $B = -0.0010823$.

Hence the particular solution is

$$Y_n = 1.0010823 \times (0.3660254)^n - 0.0010823 \times (-1.3660254)^n.$$

Compare this with the true solution

$$y_n = (0.36787944)^n.$$

The first term in Y_n is almost exactly the same as the true solution while the spurious solution has a small coefficient (-0.0010823). However see what happens as n increases: $(0.3660254)^n$ tends to zero while $(-1.3660254)^n$ is oscillating and *increasing*. Inevitably the spurious solution will eventually dominate the values obtained for Y_n. Below are the values I obtained using the recurrence relation with $Y_0 = 1$ and $Y_1 = 0.3679$.

r	0	1	2	3	4	5
x_r	0	0.1	0.2	0.3	0.4	0.5
Y_r	1	0.3679	0.1321	0.05185	0.0142	0.011725

r	6	7	8	9	10
x_r	0.6	0.7	0.8	0.9	1.0
Y_r	−0.004625	0.0104875	−0.0128	0.01804375	−0.02444375

Note that the solution decreases initially, but then the spurious solution takes over so that after $x = 0.5$ the solution is oscillating and increasing.

Solutions to the end of unit exercises in Section 6

1. The recurrence relation for the trapezoidal method is

$$Y_{r+1} = Y_r + \frac{h}{2}(Y'_r + Y'_{r+1})$$
$$= Y_r + \frac{h}{2}(Y_r + Y_{r+1}) + \frac{h}{2}(2x_r + 2x_{r+1}).$$

Hence

$$Y_{r+1}\left(1 - \frac{h}{2}\right) = Y_r\left(1 + \frac{h}{2}\right) + h(x_r + x_{r+1}).$$

(Alternatively we could have used the procedure for linear equations with $l(x) = 1$ and $k(x) = 2x$.)

With $h = 0.1$ we have

$$Y_{r+1} = \frac{1.05Y_r + 0.1(x_r + x_{r+1})}{0.95}$$

r	0	1	2	3
x_r	0	0.1	0.2	0.3
Y_r	1	1.115789	1.264820	1.450590

2. The predictor-corrector method applied to the differential equation

$$y' = 7y \qquad \text{with } y(0) = 1 \text{ and } h = 0.1$$

gives

(i) *Evaluate* $Y'_r = 7Y_r$.
(ii) *Predict* $Y^*_{r+1} = Y_r + hY'_r = Y_r + 7hY_r = 1.7Y_r$.
(iii) *Evaluate* $Y^{*\prime}_{r+1} = 7Y^*_{r+1} = 11.9Y_r$.
(iv) *Correct* $Y_{r+1} = Y_r + \dfrac{h}{2}(Y'_r + Y^{*\prime}_{r+1})$.

$$= Y_r + \frac{h}{2}(7Y_r + 11.9Y_r) = 1.945Y_r.$$

Hence $Y_{r+1} = 1.945Y_r$.

The particular solution for this recurrence relation with $Y_0 = 1$ is

$$Y_n = (1.945)^n.$$

3. For the recurrence relation

$$Y_{r+1} = Y_r + \frac{h}{5}(4Y'_r + Y'_{r+1})$$

we have

$$\phi(x_r, Y_r, Y_{r+1}, h) = \tfrac{1}{5}(4Y'_r + Y'_{r+1}).$$

Thus

$$\phi(x_r, y_r, y_r, 0) = \tfrac{1}{5}(4y_r' + y_r')$$
$$= y_r' = m(x_r, y_r).$$

Hence the method is consistent with the differential equation.

To compare this method with the trapezoidal method we compute the local truncation error. As it is an implicit method we can only compute the principal term, by assuming that all right-hand side values are correct;

i.e. $Y_{r+1} = y_r + \dfrac{h}{5}(4y_r' + y_{r+1}')$.

The Taylor series expansion for y_{r+1}' is

$$y_{r+1}' = y_r' + hy_r'' + \frac{h^2}{2}y_r''' + \cdots .$$

Hence

$$Y_{r+1} = y_r + hy_r' + \frac{h^2}{5}y_r'' + \frac{h^3}{10}y_r''' + \cdots .$$

The Taylor series expansion for y_{r+1} gives

$$y_{r+1} = y_r + hy_r' + \frac{h^2}{2}y_r'' + \frac{h^3}{6}y_r''' + \cdots .$$

$$\therefore Y_{r+1} - y_{r+1} = -\frac{3h^2}{10}y_r'' - \frac{h^3}{15}y_r''' - \cdots .$$

The principal term, $-\dfrac{3h^2}{10}y_r''$, involves h^2 while the principal term in the local truncation error for the trapezoidal method involves h^3. We would thus expect better results using the trapezoidal method.

4. For the problem

$$y' = \alpha y$$

we have

$$Y_{r+1} = Y_r + \frac{h}{5}(4\alpha Y_r + \alpha Y_{r+1}).$$

Collecting up terms gives

$$Y_{r+1}\left(1 - \frac{h\alpha}{5}\right) = Y_r\left(1 + \frac{4h\alpha}{5}\right);$$

i.e.

$$Y_{r+1} = \frac{(1 + 4h\alpha/5)}{(1 - h\alpha/5)}Y_r.$$

For absolute stability we require

$$\left|\frac{1 + 4h\alpha/5}{1 - h\alpha/5}\right| < 1.$$

If $h\alpha$ is positive $\left|1 + \dfrac{4h\alpha}{5}\right| > \left|1 - \dfrac{h\alpha}{5}\right|$ and the method is absolutely unstable.

If $h\alpha$ is negative $\left|1 + \dfrac{4h\alpha}{5}\right| < \left|1 - \dfrac{h\alpha}{5}\right|$ provided $h\alpha > -\dfrac{10}{3}$ in which case the method is absolutely stable.

Hence the method is absolutely stable if $-\dfrac{10}{3} < h\alpha < 0$.

5. Using the procedure for linear equations with $l(x) = 3$ and $k(x) = 2x$ we obtain the recurrence relation for Simpson's method applied to the linear equation as

$$Y_{r+1} = \left(\frac{4h}{1-h}\right)Y_r + \left(\frac{1+h}{1-h}\right)Y_{r-1} + \frac{2h}{3(1-h)}(x_{r-1} + 4x_r + x_{r+1}).$$

Since $x_{r-1} + x_{r+1} = 2x_r$ this can be simplified to

$$Y_{r+1} = \left(\frac{4h}{1-h}\right)Y_r + \left(\frac{1+h}{1-h}\right)Y_{r-1} + \left(\frac{4h}{1-h}\right)x_r.$$

To get started with this second-order recurrence relation, we require a value for Y_1. This can be obtained, for example, by using a one-step method with very small step-size to compute an accurate approximation to Y_1.

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